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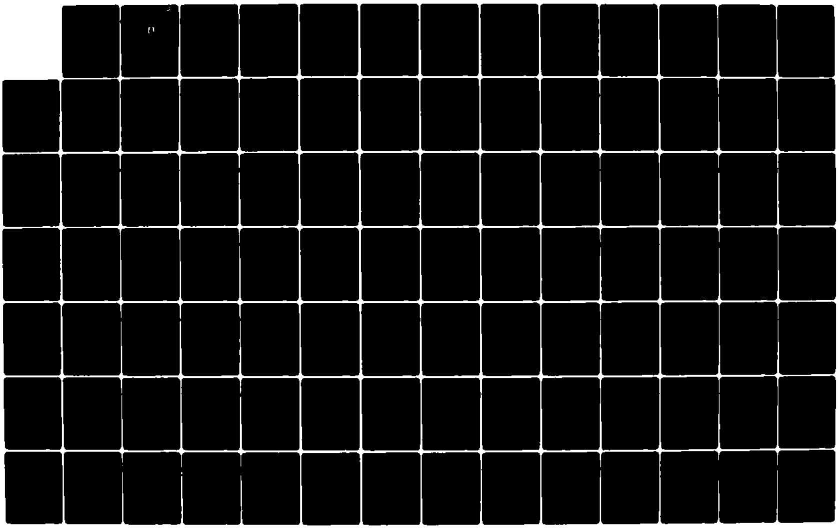

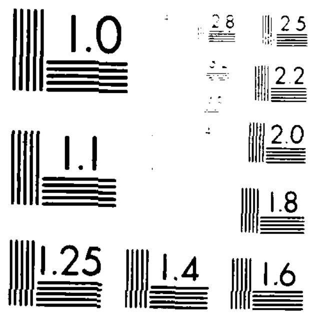
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PARTICLE DYNAMICS IN LINEAR RESONANCE ACCELERATORS

by

I.M. Kapchinskiy

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By: I.M. Kapchinskiy

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U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

*ye initially, after vowels, and after ъ, ь; e elsewhere.
When written as ё in Russian, transliterate as yё or ё.

RUSSIAN AND ENGLISH TRIGONOMETRIC FUNCTIONS

Russian	English	Russian	English	Russian	English
sin	sin	sh	sinh	arc sh	$\sinh^{-1}$
cos	cos	ch	cosh	arc ch	$\cosh^{-1}$
tg	tan	th	tanh	arc th	$\tanh^{-1}$
ctg	cot	cth	coth	arc cth	$\coth^{-1}$
sec	sec	sch	sech	arc sch	sech^{-1}
cosec	csc	csch	csch	arc csch	csch^{-1}

Russian	English
rot	curl
lg	log

GRAPHICS DISCLAIMER

All figures, graphics, tables, equations, etc. merged into this translation were extracted from the best quality copy available.

PARTICLE DYNAMICS IN LINEAR RESONANCE ACCELERATORS.

I. M. Kapchinskiy.

Page 2 No Typing.

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INTRODUCTION.

The linear accelerators of the charged/loaded atomic particles at present increasingly more widely are applied in experimental physics and in some areas of technology. This caused the considerable progress in the development both the theoretical problems, connected with particle dynamics in the linear accelerators and the engineering questions, connected with the constructions/designs of the accelerating systems and auxiliary technological equipment. Basic advantage of linear accelerators - simplicity of the beam extraction of the accelerated particles. Linear accelerator makes it possible to obtain the well collimated beam with the relatively greater instantaneous intensity. One should note also that the linear accelerators, as a rule, are reliable. Therefore they proved to be the most adequate/approaching type of injector for the powerful/thick proton synchrotrons. However, linear accelerators are convenient also as the independent accelerators, since they can work in the continuous duty and give in this case beam with the large average/mean intensity.

Upon the acceleration of ions to the energies, which exceed 5 MeV, usually is utilized high-frequency accelerating field, which makes it possible to pass the particles through the large number of accelerating gaps and, therefore, to restrict the maximum values of stresses/voltages in the accelerator. High energy particles with the given speed, which approaches unity, can be accelerated in the field of the traveling wave. In the high-frequency field the beam decomposes into the clusters. The frequency of bunches coincides with the frequency of accelerating voltage. Therefore such accelerators are called, according to the steady terminology, resonance.

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In the linear resonance accelerators always can be isolated the fundamental harmonic of the component of accelerating field, which is spread in the direction of acceleration. As it will be shown, precisely, this harmonic gives the basic contribution to the particle acceleration. In traveling-waves accelerator is formed/shaped only this harmonic. In the general case accelerating field contains wide the spectrum of running harmonics; the fundamental harmonic, isolated above, is called the equivalent traveling wave. Linear resonance accelerators, from the point of view of particle dynamics in accelerating field, can be divided into the accelerators of two types: in first type accelerators the phase speed of the equivalent traveling wave is lower than the speed of light; in second type accelerators the phase speed of the equivalent traveling wave is equal to the speed of light. In the first case phase wave velocity monotonically grows/rises, and the longitudinal velocities of particles vary relative to the instantaneous value/significance of phase speed. The secondly longitudinal particle motion is threshold. These differences in the particle motion determine many special features/peculiarities of each type of accelerator (capture of particles in acceleration mode, formation of clusters, etc.). First type linear accelerators are utilized for accelerating the heavy particles whose given speed remains substantially smaller unity up to the exit energy. The initial sections of the electron accelerator (in which occurs the grouping and the preliminary dispersal/acceleration of electrons) are also first type accelerators. However, due to a small rest mass electrons acquire the speeds, which approach the speed of light, even with the energy several mega-electron-volts, and further acceleration of electrons is conducted in second type accelerator.

With an increase in the energy of particles is facilitated the beam focusing. This is connected, in the first place, with the decrease of the angular scatter of the trajectories of the accelerated particles, and in the second place, with the decrease of the Coulomb pushing apart of particles in the beam. It is natural that questions of focusing prove to be most serious precisely during the construction of first type linear accelerators, which leads to the considerable complication of the theory of particle motion in these accelerators.

Subsequently let us examine in essence first type linear resonance accelerators, intended for accelerating the ions. However, in the appropriate places will be completed transition to the maximum permissible value/significance of phase speed $v_\Phi = c$.

One of the most important characteristics of beam of particles, injected into the linear accelerator, is its phase volume.

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The phase volume of beam is determined by the disordered scatter of particles on the attitude and by speeds. In accordance with Liouville's theorem the phase volume is a value invariant. Liouville's theorem is valid under specific conditions which with a sufficient accuracy are always satisfied in the linear accelerators. Any linear accelerator can take and accelerate beam of particles, limited by certain finite quantity of phase volume. This value is determined by the capacity of this accelerator. In practice the phase volumes of the beams, generated by different ion sources, prove to be comparable in the value with the capacity of accelerators or even exceed it. The fact indicated leads to the specific problems which must be solved during the design of ionic sources and linear accelerators.

Particle motion in the accelerating and focusing fields is connected both with the action of the proper field of beam and with the disordered scatter of thermal particle speed, the determined phase volume of beam. If it is disregarded by both factors, it proves to be sufficient to examine the motion of one particle in the assigned applied fields. However, already accumulated experiment design and operation of contemporary linear accelerators showed that the factors indicated should be considered. If particle density in the phase space of beam is sufficiently great, it is possible to disregard the scatter of thermal particle speed. But if phase density is small, and the phase volume, occupied with the particles of beam, is relatively great, admissibly disregard/neglect the pushing apart Coulomb forces. In many practically important cases it is necessary to consider with the fact that effects of both factors mentioned above are commensurated. Neglect of the space charge of beam or by the scatter of thermal velocities can lead to the significant errors during the calculation of the parameters of accelerator.

An increase in the intensity of the accelerated beams in proton synchrotrons and respectively in the linear accelerators - injectors - is at present one of the urgent problems of accelerative technology. Therefore the problems of the acceleration of beams with the high pulse intensity were especially intensely studied experimentally and theoretically into the latter several years both in the USSR and abroad.

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Now already there are monographs with the systematized presentation of questions of particle dynamics in linear resonance accelerators [1-3], but the theory of intense beams, connected with the simultaneous account of final phase volume and final beam current, even on I could find a sufficient reflection. Goal of this book - to a certain extent to complete a deficiency/lack in the literature on the questions indicated. In the first two chapters is examined particle dynamics in the beams with the negligible current density. The material of these chapters is constructed so as to first isolate the basic questions with which we encounter in the examination of beams with the essential space charge.

In contrast to the cyclic ones in the linear particle accelerators they are accelerated for a small time interval and acquire high energy on the relatively short path. In view of the short time of interaction of particle with the field in the linear accelerator different resonance effects, which lead to the loss of beam in the cyclic accelerators, have comparatively low value/significance in the theory of linear accelerators. On the other hand, due to the rapid set of energy in the linear accelerator becomes very essential the effect of defocusing of particles by accelerating field. These special features/peculiarities lead to the fact that the theory of particle motion in the cyclic accelerators both in the examination of the beams of zero intensity and during the analysis of the dynamics of intense beams.

The author hopes that the book will prove to be useful and to specialists, who do not work in the area of accelerative technology, but by the problems of the beam shaping of high intensity interesting in other areas of technology.

The author expresses deep gratitude to D. G. Koshkarev and V. K. Plotnikov for the valuable discussions, and also to B. I. Bondaryev and K. I. Guseva for the composition of the abstract of lectures, placed as the basis of the book.

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Chapter 1.

Longitudinal vibrations of particles in beams with the negligible density of space charge.

§1.1. Dynamics of the synchronous particle, which moves along the axis of accelerator.

Let the charged particle beam be accelerated in the field of the running simple harmonic wave with the longitudinal component

$$E_z = E\cos\left(\omega t - \int_0^z k(z)\,dz\right). \qquad (1.1)$$

Henceforth we will use the Cartesian coordinate system with z axis, directed along the axis of linear accelerator. Phase wave velocity v_Φ we will consider the assigned monotonically increasing function of the longitudinal coordinate z:

$$k(z) = \frac{\omega}{v_\Phi(z)}. \qquad (1.2)$$

Let us assume $v_\Phi < c$. Propagation of the traveling wave with the phase speed, the lower speed of light/world, possibly, in particular, in the waveguide, loaded with appropriate diaphragms [1, 4-6]. Let us isolate in the beam certain particle whose speed at each moment of time coincides with phase wave velocity, which accompanies particle. This particle, moving at the current point of wave with certain fixed/recorded phase φ_s, it will test/experience the action of the longitudinal force

$$F_z = eE\cos\varphi_s.$$

In the conformity with expression (1.3) phase φ_s is counted off from the moment of time with which the field at the particular point reaches maximum value/significance. Let us select for certainty e>0. Since particle acceleration is determined by force F_z, for each given value $\cos\varphi_s > 0$ it can be selected the amplitude of the traveling wave E (z), which ensures the current equality to phase wave velocity and longitudinal velocity of particle, in other words, that ensures the retention/preservation/maintaining assigned phase φ_s.

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The particle whose speed coincides at each moment of time with phase wave velocity, is called synchronous particle, and the fixed value of phase φ_s — by synchronous phase.

It is obvious that in the extreme case of acceleration in the traveling wave when $v_\Phi = c$, there is no synchronous particle in the

beam.

Let us examine the case of accelerating the beam of particles in the field of standing waves. The high-frequency system of linear accelerator with the standing waves consists of the consecutive accelerating gaps and, therefore, it has almost periodic structure. In the simplest case each period of structure contains one accelerating gap. The parameters of the periods of the accelerating structure monotonically are changed along the longitudinal axis of accelerator. For the proton linear accelerators in the range of energies from 10 to 100 MeV is predominantly utilized the high-frequency system, for the first time used on the accelerator in Berkeley [7]. This system is the hollow cylindrical cavity, loaded with drift tubes. Particles are accelerated in the clearance between drift tubes and are shielded from the field (to the period when field it has opposite direction) in the channels, arranged/located within the tubes. Fig. 1.1a, schematically depicts the period of the acceleration of high-frequency system with drift tubes. L_n — length of the n period of acceleration, equal to the distance between the middles of adjacent drift tubes; g_n — length of the accelerating clearance. The periods of some other accelerating systems are schematically given in Fig. 1.1, b-e. The accelerating device/equipment in Fig. 1.1b, consists of cylindrical capacitor/condenser with two accelerating clearances, which adjoin the grounded rings. Fig. 1.1~~b~~c, d gives the diverse variants of the use of the retarding spirals within metal tube [8].

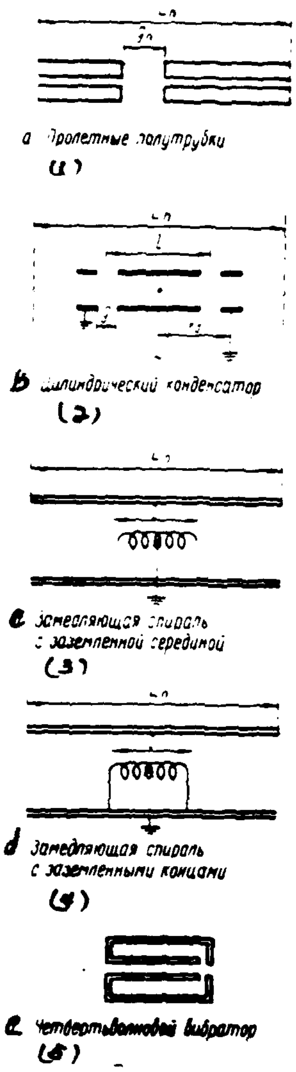

Fig. 1.1. Key: (1). drift half-tubes. (2). Cylindrical capacitor/condenser. (3). Retarding spiral with grounded middle. (4). Retarding spiral with grounded ends/leads. (5). Quarter-wave antenna.

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In Fig. 1.1e field in the accelerating clearance is formed/shaped with the aid of the cavity quarter-wave resonator. The accelerating devices/equipment, shown in Fig. 1.1b, e,, until now, are utilized only in the circular accelerators. All known at present electronic linear accelerators are the waveguides, loaded with disks [6, 9-11]. In the accelerating systems of contemporary proton accelerators are commonly used drift tubes [12-17].

Questions of radio-frequency technology of linear resonance accelerato : exceed the limits of the book. Therefore we will not discuss comparative advantages and disadvantages those in different accelerating structures, and also we will not examine new developments in this region. Some questions, which relate to the accelerating devices/equipment technique, are briefly examined in work [2] and in other given sources. Let us note that further presentation of the theory of particle motion in linear resonance accelerators can be attributed with some unprincipled changes to any accelerating structure. for the concreteness subsequently we will have in mind the accelerating system with drift tubes.

The longitudinal component of high-frequency electrical field on the axis of accelerating gap takes the form

$$E_z(z,\ t) = E_g(z)\cos\omega t. \qquad (1.4)$$

Here $E_z(z)$ — law of amplitude distribution of the longitudinal component of the field of standing wave along the axis in the n period of the accelerating structure. We will call particle in the field of the standing wave of synchronous, if it flies certain fixed/recorded point of each period of structure with the same phase of high-frequency field. This point is called the electrical center of period [7]. Let z_{n-1}, z_n — coordinate of beginning and end/lead of the n period of the accelerating structure. An energy gain of any particle, which moves along the axis, at the length of period is determined by the expression

$$\Delta W = W_n - W_{n-1} = e \int_{z_{n-1}}^{z_n} E_z(z) \cos \omega t(z)\, dz, \qquad (1.3)$$

where W - kinetic energy of the particle: $W=(m-m_0)c^2$. Will select certain point z_0 within the period and will fix phase φ_s with which synchronous particle is passed this point. Regarding the synchronous phase, value/significance φ_s is retained one and the same in the electrical center of each period of the accelerating structure. The distance between the electrical centers of the adjacent periods of structure synchronous particle flies for the time, multiple to the period of high-frequency field (with the cophasal supply of accelerating gaps).

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For the synchronous particle we have

$$t(z)=\frac{\varphi_s}{\omega}-\int_{z_0}^{z}\frac{d\zeta}{v_s(\zeta)}, \qquad (1.6)$$

where v_s — current longitudinal velocity of synchronous particle. After substituting equality (1.6) into expression (1.5), we will obtain

$$\Delta W_s=e\cos\varphi_s\int_{z_{n-1}}^{z_n}E_g(z)\cos\frac{2\pi}{\lambda}\int_{z_0}^{z}\frac{d\zeta}{\beta_s(\zeta)}\,dz-$$

$$-e\sin\varphi_s\int_{z_{n-1}}^{z_n}E_g(z)\sin\frac{2\pi}{\lambda}\int_{z_0}^{z}\frac{d\zeta}{\beta_s(\zeta)}\,dz, \qquad (1.7)$$

where λ - wavelength of accelerating field in the free space; β - given particle speed. The predominant effect of the components/terms/addends in the right side of expression (1.7) depends on the parity of function $E_g(z)$ relative to point z_0. If function $E_g(z)$ is close to the even (which corresponds to the accelerating devices/equipment in Fig. 1.1a, ~~b~~ c, e), then predominates the first term. But if function $E_g(z)$ is close to the odd (see Fig. 1.1 b, ~~c~~ d), then predominates the second term. Let us assume that the basic contribution to an energy gain gives the first term. Then expression (1.7) is conveniently represented in the form

$$\Delta W_s=e\cos\varphi_s L_n\frac{1}{L_n}\int_{z_{n-1}}^{z_n}E_g(z)\,dz\,T,$$

where

$$T=\frac{\int_{L_n}E_g(z)\cos\frac{2\pi}{\lambda}\int_{z_0}^{z}\frac{d\zeta}{\beta_s}\,dz}{\int_{L_n}E_g(z)\,dz}-\operatorname{tg}\varphi_s\frac{\int_{L_n}E_g(z)\sin\frac{2\pi}{\lambda}\int_{z_0}^{z}\frac{d\zeta}{\beta_s}\,dz}{\int_{L_n}E_g(z)\,dz}. \qquad (1.8)$$

The value, determined by expression (1.8), is called the factor of transit time. This factor depends substantially on the geometry of field in the accelerating gap and on the particle speed. As will be shown subsequently, the dependence of the factor of transit time on the value of synchronous phase, clearly entering in the second member of expression (1.8), and implicitly concealed/latent also in the first member, it is unessential and in the first approximation, disappears.

Value

$$E_0 = \frac{1}{L_n} \int_{z_{n-1}}^{z_n} E_g(z)\,dz \tag{1.9}$$

let us name/call middle field on the axis of accelerator. This value is equal to the amplitude of the longitudinal component of accelerating field, accelerating structure averaged on the period.

Taking into account designations (1.8), (1.9) we will obtain simple expression for an energy gain of synchronous particle in the period of the accelerating structure

$$\Delta W_s = eE_0TL_n\cos\varphi_s. \tag{1.10}$$

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Simplicity of this expression is explained only by the fact that all difficulties of calculating interaction of particle with the assigned high-frequency field are transferred to the determination of the factor of transit time. from initial expressions (1.4), (1.6) it follows that the phase of field, in which the particle falls substances the electrical center of period, must be counted off from the moment/torque of the maximum of field.

Let us name/call first approximation to a factor of transit time value

$$T_0 = \frac{1}{E_0 L_n} \int_{L_n} E_z(z) \cos \frac{2\pi (z - z_0)}{\lambda \beta_{cp}} dz. \qquad (1.11)$$

where β_{cp} is determined by the equality

$$\beta_{cp} = \frac{1}{2} (\beta_{n-1} + \beta_n). \qquad (1.12)$$

and β_n — given speed of synchronous particle at the end of the n period of the accelerating structure. According to expression (1.11), during the calculation of first approximation to a factor of transit time for the field with the predominant even component (relative to electrical center) the second term in expression (1.8) is thrown/rejected, and particle speed in the period is assumed the constant, equal to β_{cp}. If is assigned middle field on the axis, then into equality (1.11) enters the first approximation to length of period, determined by the expression

$$L_{n_0} = k \beta_{cp} \lambda. \qquad (1.13)$$

Value k is called the multiplicity of the period of acceleration. With the cophasal supply of accelerating gaps value k determines the number of periods of high-frequency field, for which synchronous particle flies the distance between the electrical centers of the adjacent periods of structure. Repeatedly were proposed different high-frequency systems with the noncophasal supply of accelerating gaps. noncophasal supply is possible, if accelerating gaps are electrically untied, for example, if adjacent periods in the resonator, loaded with drift tubes, are divided by continuous metallic partitions/baffles. In the simplest of such systems $L_n = L = \mathrm{const}$. A similar system can be treated as system with the variable/alternating multiplicity of the accelerating structure

$$k = \frac{L}{\beta_{cp}\lambda}.$$

or as system with the assigned multiplicity, but with a monotonically decreasing equivalent wavelength of

$$\lambda_{экв} = \frac{L}{k\beta_{cp}}.$$

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With the assigned multiplicity k the first approximation to a factor of transit time can be represented in the form

$$T_0 = \frac{1}{E_0} \cdot \frac{1}{L_n} \int\limits_{L_n} E_g(\zeta) \cos \frac{2\pi k}{L_n} \zeta \, d\zeta = \frac{A_k}{2E_0}, \qquad (1.14)$$

where A_k — amplitude of the k cosinusoidal harmonic of the three-dimensional/space resolution by Fourier of high-frequency field. Formula (1.14) is convenient, if field distribution of standing wave in the accelerating gap they calculate, on the basis of the appropriate boundary-value problem.

When the sizes/dimensions of the accelerating device/equipment are small in comparison with the wavelength of high-frequency field, it is convenient to utilize for calculating the factor of transit time the semi-empirical method, based on the electrostatic approximation/approach to goal [18]. In the electrostatic approximation/approach potential distribution in accelerating gap $V(\zeta)$, where $\zeta = z - z_0$, it is possible to measure by simulation of gap/interval with the aid of electrolytic bath. Since during calculations of the factor of transit time are utilized the experimental data, for decreasing the errors of calculation the desirable curved of potential distribution not to differentiate, but to integrate. Taking integral (1.11) in parts and taking into account that $E_g(\zeta) \, d\zeta = dV$, we obtain

$$T_0 = (-1)^k + \frac{2\pi k}{V_0 L_n} \int\limits_{L_n} V(\zeta) \sin \frac{2\pi k}{L_n} \zeta \, d\zeta, \qquad (1.15)$$

where V_0 - complete potential drop along the axis of accelerating gap.

For calculating the parameters of the accelerating system it suffices to utilize first approximations to factor of transit time and length of the period of structure. Actually/really, it is possible to show that the corrections of the second approximation/approach in the middle fields in practice utilized in the ionic linear accelerators prove to be negligible. These corrections are caused by the inconstancy of the speed of synchronous particle for the period of structure and by odd component of the distribution of standing wave in the accelerating gap (in the systems in Fig. 1.1, a, c, e). It is obvious that the position of electrical center should be selected in such a way, as to maximally decrease odd component of distribution (in the systems in Fig. 1.1.a, c, e) or even component (in the systems in Fig. b, c). In work [7] is examined the selection of the position of electrical center for the accelerating system with drift tubes. In this work it is proposed to select the electrical center of each accelerating clearance z_0 in such a way that the second member of expression (1.8) would become zero

$$\int\limits_{L_n} E_g(z-z_0)\sin\frac{2\pi}{\lambda}\int\limits_0^{z-z_0}\frac{d\zeta}{\beta_s(\zeta)}\,dz=0. \qquad (1.16)$$

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But the coordinate of electrical center during this determination must be calculated, moreover its position is changed with a change in the synchronous phase. Since the allowances for the longitudinal arrangement of drift tubes relate to the coordinates of electrical centers, this definition, which formally simplifies the calculation of corrections of second approximation/approach, in practice proves to be inconvenient. To it is simpler select the electrical center of period in the geometric center of the accelerating clearance and to treat the second member of expression (1.8) as the supplementary correction of the second approximation/approach. Both corrections prove to be the values of one order. When the correction of the second approximation/approach can be disregarded/neglected, the geometric center of clearance coincides with the electrical center, determined by equality (1.16). Analogous considerations occur, also, during the selection of electrical centers in the accelerating systems of other types. In Fig. 1.1 electrical centers are noted by asterisks.

Let us rate/estimate the corrections of the second approximation/approach. The current particle speed in the period of the accelerating structure can be represented by the series/row

$$\beta(z)=\beta(z_0)+\frac{d\beta}{dz}(z_0)(z-z_0)+\frac{1}{2}\frac{d^2\beta}{dz^2}(z_0)(z-z_0)^2+\ldots \quad (1.17)$$

Rate of change in the longitudinal component of impulse/momentum/pulse is determined by the expression

$$\frac{dp_z}{dt} = eE_g(z)\cos\omega t(z).$$

Since

$$\frac{d\beta}{dt} = \frac{1}{m_0 c\gamma^3}\cdot\frac{dp_z}{dt}, \qquad (1.18)$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ — Lorenz's factor,

$$\frac{d\beta}{dz}(z) = \frac{eE_g(z)}{\beta\gamma^3\mathcal{E}_0}\cos\omega t(z); \qquad (1.19)$$

$\mathcal{E}_0 = m_0c^2$ — rest energy of particle. The higher derivatives are calculated by differentiation of expression (1.19). With substitution $z=z_0$ one should consider that for synchronous particle $\omega t(z_0)=\varphi$. For evaluating the corrections of the second approximation/approach let us assume: $E_g = \text{const}$ with $|z-z_0|<g/2$ and $E_g = 0$ with $|z-z_0|>g/2$. This approximation of field in the accelerating gap is occasionally referred to as square wave. After substituting series/row (1.17) into expression (1.8) and after producing integration, we will obtain

$$T = T_0\left(1 - \frac{k}{6}\chi\alpha\right), \qquad (1.20)$$

where T_0 is determined by formula (1.11) and does not require the approximation of field by square wave.

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Low parameter χ is assigned by the expression

$$\chi = \frac{\pi e E_0\lambda \sin\varphi_s}{2\beta_n\gamma_n^3\mathcal{E}_0}, \qquad (1.21)$$

where $\beta_n = \beta_s(z_0)$ — speed of synchronous particle in the electrical center

$$\beta_u = \beta_{cp}\left(1 - \frac{k}{2}\chi\alpha\right). \tag{1.22}$$

Value

$$\alpha = \frac{g}{\lambda\beta_u} \tag{1.23}$$

is called the coefficient of clearance.

The length of the period of the accelerating structure, strictly speaking, is determined by the equality

$$\int_0^{L_n} \frac{dz}{\lambda\beta_s(z)} = k.$$

After using series/row (1.17), we will obtain

$$L_n = L_{n_0}\left(1 - \frac{1}{3}\chi\alpha^2\right). \tag{1.24}$$

The corrections of the second approximation/approach can be disregarded/neglected, if parameter χ is sufficiently low.

One of the most important parameters of linear accelerator is specific acceleration W_λ, equal to the relation of an energy gain of synchronous particle at the length λ to the rest energy. In the fields of the standing waves

$$W_\lambda = \frac{eE_0T\lambda}{\mathscr{E}_n}\cos\varphi_s, \tag{1.25}$$

in the field of the traveling wave

$$W_\lambda = \frac{eE\lambda}{\mathscr{E}_0}\cos\varphi_s. \tag{1.25a}$$

Parameter χ can be expressed through the specific acceleration

$$\chi = \frac{\pi \, \mathrm{tg}\,\varphi_s}{2\gamma^3 T} \cdot \frac{W_\lambda}{\beta}. \qquad (1.21a)$$

Usually $T \sim 1$, $\mathrm{tg}\,\varphi_s \leqslant 1$. Thus, the smallness of parameter χ is provided, if specific acceleration is small in comparison with the instantaneous value of the given particle speed. Specific acceleration very is simply connected with a partial increase in the impulse/momentum/pulse of synchronous particle in the period of the accelerating structure. If partial increases in the impulse/momentum/pulse and energy are small, then, as it is easy to show,

$$\frac{\Delta p_s}{m_0 c} = \frac{\Delta W_s}{\beta_s \mathscr{E}_0}.$$

Substituting $\Delta W_s = \frac{W_\lambda \mathscr{E}_0}{\lambda} L_n$, we obtain

$$\frac{\Delta p_s}{m_0 c} = k W_\lambda. \qquad (1.26)$$

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Nonrelativistic approximation/approach formula (1.26) is reduced to the form

$$\Delta\beta_s = k W_\lambda. \qquad (1.27)$$

The quantitative estimations, which base the possibility of the deletion of the corrections of the second approximation/approach, will be given in §1.4.

Given formulas (1.8), (1.10), (1.11), (1.14), (1.16) are modified, if in field distribution $E_z(z-z_0)$ predominates odd component. For the orientation let us give the approximation formulas for the factor of transit time and partial increase in energy in some accelerating devices/equipment, represented in Fig. 1.1. Synchronous phase everywhere is counted off from the moment of time with which the standing wave reaches amplitude value.

1. Drift tubes (see Fig. 1.1a),

$$T=\frac{1}{I_0\left(\frac{\pi d}{\beta\lambda}\right)}\cdot\frac{\sin\pi\alpha}{\pi\alpha};\quad \Delta W_s=eV_0T\cos\varphi_s. \qquad (1.28)$$

where I_0 - modified Bessel function of zero-order; d - diameter of aperture opening/aperture in drift tube; V_0 - amplitude value of potential difference between drift tubes.

2. Cylindrical capacitor/condenser (see Fig. 1.1b),

$$T=\frac{1}{I_0\left(\frac{\pi d}{\beta\lambda}\right)}\cdot\frac{\sin\pi\alpha}{\pi\alpha}\sin\frac{\pi l}{\beta\lambda};\quad \Delta W_s=-2eV_0T\sin\varphi_s. \qquad (1.29)$$

where d - inner diameter of capacitor/condenser; α - coefficient of clearance between capacitor/condenser and each grounded ring; l -

distance between geometric centers of clearances; V_0 - amplitude value of potential difference between capacitor/condenser and each of grounded rings.

3. Retarding spiral with grounded middle (see Fig. 1.1c).

$$T=\frac{\frac{2l}{\beta\lambda}}{1-\left(\frac{2l}{\beta\lambda}\right)^2}\cos\frac{\pi l}{\beta\lambda};\ \Delta W_s=\frac{2}{\pi}eElT\cos\varphi_s. \quad (1.30)$$

4. Retarding spiral with grounded ends/leads (see Fig. 1.1d).

$$T=\frac{\frac{2l}{\beta\lambda}}{1-\left(\frac{2l}{\beta\lambda}\right)^2}\cos\frac{\pi l}{\beta\lambda};\ \Delta W_s=-\frac{2}{\pi}eElT\sin\varphi_s. \quad (1.31)$$

In formulas (1.30), (1.31) l - axial length of spiral; E - amplitude of the longitudinal component of field in the antinode.

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In conclusion let us examine one of the possible procedures of calculation of the periods of high-frequency structure for the accelerator with drift tubes. For the calculation initial is the dependence of the factor of transit time on the instantaneous value of the given speed of the synchronous particle $T=T(\beta)$. This dependence in the electrostatic approximation/approach is determined by formula (1.15) and can be interpolated by the series/row of the discrete/digital values T with the appropriate smoothing of the curve, passing through experimental points. Let t_s, z_s — instantaneous values of the flight time and coordinate of synchronous particle. Then

$$dz_s = \frac{dW'_s}{w_0}; \quad dt_s = \frac{dz_s}{c\beta_s}.$$

where

$$w_0 = \frac{dW'_s}{dz_s} = eE_0 \cos\varphi_s T(\beta).$$

Since $W'_s = (m - m_0)c^2$, that

$$dW'_s = \mathcal{E}_0 \beta (1-\beta^2)^{-\frac{3}{2}} d\beta.$$

During the calculation let us assign the parameter

$$\Lambda = \frac{eE_0\lambda}{\mathcal{E}_0} \cos\varphi_s, \qquad (1.32)$$

of those connected with the specific acceleration with the equality

$$W'_\lambda = \Lambda T(\beta). \qquad (1.33)$$

Then

$$dz_s = \frac{\lambda}{\Lambda} \cdot \frac{\beta d\beta}{(1-\beta^2)^{3/2} T(\beta)};$$

$$dt_s = \frac{\lambda}{c\Lambda} \cdot \frac{d\beta}{(1-\beta^2)^{3/2} T(\beta)}.$$

Let us introduce the new dimensionless variables

$$\xi_s = \frac{\Lambda}{\lambda} z_s; \quad \tau_s = \frac{c\Lambda}{\lambda} t_s.$$

In these variable/alternating

$$\tau_s = \int_{\beta_0}^{\beta} \frac{d\beta}{(1-\beta^2)^{3/2} T(\beta)}; \quad \xi_s = \int_{\beta_0}^{\beta} \frac{\beta d\beta}{(1-\beta^2)^{3/2} T(\beta)},$$

where β_0 - given speed of injection. By latte last integrals in the parametric form is assigned dependence $\xi_s = f(\tau_s)$ which can be constructed graphically (Fig. 1.2) or written n the form of table. Let us divide the axis of abscissas into equa intervals $\Delta\tau_s = k\Lambda$.

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To space $\Delta\tau_s = k\Lambda$ corresponds $\Delta t_s = k\frac{2\pi}{\omega}$, i.e. tim of flight between the electrical centers of adjacent periods. The o linates of dividing points determine the coordinates of the begin .ng of each n period of the accelerating structure. The lengths of dr ft tubes are calculated from the assigned dependence $\alpha(\beta)$, which ensu es the equality of the natural frequencies of all periods of structu e.

The periods of the accelerating structur and length of drift tubes can be calculated also consecutively/se ially, utilizing formulas (1.13), (1.11), (1.10). Since for ea n following period of structure to us is known the inlet velocity while into formulas (1.11), (1.13) enters speed β_{cp}, average/mean n the period then the latter must be calculated either by the methc of successive approximations or, is more rough, with the ai of the approximate equality

$$\beta_{cp} \approx \beta_{n-1} + \frac{k\Lambda}{2\gamma_{n-1}^3} T(\beta_{n-1}).$$

§1.2. Phase stability of particles. Capture of particles into acceleration mode.

The particles, injected into first type linear resonance accelerator, are seized into acceleration mode because of the process of phase stability [19, 20]. The proce·· of phase stability leads to the fact that the particles, which satisfy the specified initial conditions, stably are grouped in the area of the synchronous phase of high-frequency field. The remaining particles, which do not fall into the stable region about the synchronous phase, on the average do not extract energy in high-frequency field and retire from the game. Therefore beam decomposes lengthwise to the clusters, which follow after each other with the frequency of accelerating field. The elementary picture of phase stability it is easy to explain based on the example of particle acceleration in the traveling wave. In this case the region, occupied with the accelerated particles, is located about the synchronous phase on a decrease in the crest of wave (Fig. 1.3a). Actually/really, let us examine the particles, which move sufficiently closely to the synchronous. If any particle has a speed, the lower speed of synchronous particle, then it will lag behind the synchronous and it will hit the region of the traveling wave with the increased strength of field. This particle will undergo acceleration, which exceeds travelling-wave acceleration; therefore its speed will grow/rise relative to wave, until particle passes synchronous phase.

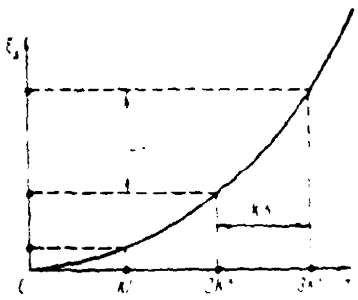

Fig. 1.2.

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Particle with the initial velocity of of higher than phase wave velocity, will anticipate/lead synchronous particle and it will hit the region of the traveling wave with the lowered/reduced strength of field. The speed of this particle relative to phase wave velocity will decrease, until particle falls behind the synchronous phase. Thus, the particles, which fall into the accelerator in the nonsynchronous phase of the traveling wave, but it is sufficiently close to the synchronous phase, they prove to be in potential well and complete longitudinal vibrations relative to synchronous phase. In the average speed of nonsynchronous particles proves to be equal monotonically growing phase wave velocity. Entire bunch of particles, seized in the mode of acceleration in this period of high-frequency field, will move with the speed, equal at each moment of the time of phase wave velocity. As it will be shown subsequently, the spread/scope of phase oscillations decreases with an increase in the energy of particles.

Similar pattern occurs upon the particle acceleration in the field of standing waves. Stable synchronous phase falls to the period of the increase of field in the accelerating clearance (Fig. 1.3b). The particle, which moves behind the synchronous, will obtain in the accelerating clearance a larger partial increase in energy, than synchronous, while the particle, which moves in front of the synchronous, is smaller. The phase of the standing wave with which the given particle, seized into acceleration mode, passes the electrical centers of the consecutive clearances, will oscillate around the synchronous phase.

The longitudinal vibrations of particles in the traveling wave are described by the differential equations whose solutions determine region of capture of particles in acceleration mode, and also frequency and damping of oscillations/vibrations. In the linear accelerators with the standing waves the longitudinal vibrations, strictly speaking, are described not by differential equations, but by difference equations. But if a change in the phase of the flight/span of electrical center and a partial increase in energy in each period of the accelerating structure is sufficiently small, then finite increments can be replaced by differentials, which actually reduces the goal to the particle acceleration in the equivalent traveling wave. Quantitative estimations show that with usually utilized in the linear ion accelerators middle fields this replacement does not lead to the significant errors in calculations.

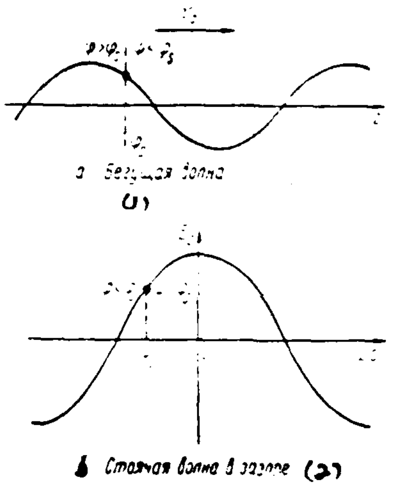

Fig. 1.3. Key: (1). Traveling wave. (2). Standing wave in clearance.

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For evaluating the permissible errors during the replacement of difference equations of differential equations let us compare solutions of both systems of equations. This comparison it suffices to conduct for the simplest case, disregarding relativistic relationships/ratios and damping of longitudinal vibrations. A difference in solutions of both systems of equations is greatest with the low energies of particles.

Let us examine for concreteness the accelerating system with drift tubes (Fig. 1.4). Let v_{ns}, v_n — speed of synchronous and nonsynchronous particles respectively within n drift tube; φ_n – phase of field, in which nonsynchronous particle flies the electrical center of the accelerating clearance before n drift tube. Let us designate

$$\psi_n = \varphi_n - \varphi_s;$$
$$g_n = \frac{v_n - v_{ns}}{v_{ns}}. \qquad (1.34)$$

For the particles, little which are deflected from the synchronous particle, we have $\psi_n \ll 1;\ g_n \ll 1$. The raid of a phase difference in one period of the accelerating structure is equal to

$$\Delta\psi_n = \psi_{n+1} - \psi_n = \omega(\Delta t_n - \Delta t_{ns}).$$

where Δt_n, Δt_{ns} – time of landing run of the nonsynchronous and synchronous particles between the electrical centers of the n-th and (n+1)-th clearances

$$\Delta t_{ns} = k\frac{2\pi}{\omega}; \quad \Delta t_n = \Delta t_{ns}\frac{v_{ns}}{v_n}.$$

Hence

$$\Delta\psi_n \approx -2\pi k g_n. \qquad (1.35)$$

On the other hand,

$$\Delta g_n = g_{n+1} - g_n = \frac{v_{n+1} - v_{n+1,s}}{v_{n+1,s}} - \frac{v_n - v_{ns}}{v_{ns}}.$$

Let us place $v_{n+1,s} \approx v_{ns}$, this equivalently, as it is possible to show, to failure of the examination of the process of damping the longitudinal vibrations

$$\Delta g_n = \frac{\Delta v_{ns}}{v_{ns}}\left(\frac{\Delta v_n}{\Delta v_{ns}} - 1\right).$$

Repeating linings/calculations of §1.1, for a partial energy gain of nonsynchronous particle we will obtain the expression, analogous to formula (1.10):

$$\Delta W_n = W_{n+1} - W_n = eE_0L_nT\cos\varphi_{n+1}.$$

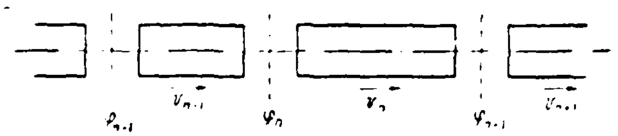

Fig. 1.4.

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As was shown in §1.1, the factor of transit time T in the first approximation, does not depend on the phase of particle. With an accuracy to equality $v_n = v_{ns}$ value T is identical for the synchronous and nonsynchronous particles. In nonrelativistic approximation/approach $W_n = \frac{1}{2} m_0 v_n^2$; whence

$$\Delta W_n \approx m_0 v_{ns} \Delta v_n;\ \Delta W_{ns} \approx m_0 v_{ns} \Delta v_{ns}$$

and

$$\Delta g_n = \frac{\Delta v_{ns}}{v_{ns}} \left(\frac{\cos \varphi_{n-1}}{\cos \varphi_s} - 1 \right) \approx -\frac{\Delta v_{ns}}{v_{ns}} \operatorname{tg} \varphi_s \psi_{n-1}.$$

Substituting expression (1.27) into the latter/last equality and introducing for the brevity a designation

$$m^2 = -2\pi k^2 \frac{W_\lambda \operatorname{tg} \varphi_s}{\beta_{ns}}, \qquad (1.36)$$

we have

$$\Delta g_n = \frac{m^2}{2\pi k} \psi_{n+1}. \qquad (1.37)$$

Equalities (1.35) and (1.37) are the equations of small longitudinal vibrations in the first differences. Let us compose second difference $\Delta\psi_n - \Delta\psi_{n-1}$. Utilizing expressions (1.35), (1.37), we obtain the following equation of longitudinal vibrations for variable/alternating ψ_n:

$$\Delta\psi_n - \Delta\psi_{n-1} - m^2 \psi_n = 0. \qquad (1.38)$$

Before solving equation we reduce it to a differential equation, replacing the finite increments by differential.

In equation (1.38), ~~(1.38)~~ n - number of periods of the accelerating structure, passed with particle from the beginning of accelerator. Since Δn=1,

$$\Delta\psi_n = \frac{\psi_{n+1}-\psi_n}{\Delta n} = \frac{d\psi}{dn};\ \Delta\psi_n - \Delta\psi_{n-1} = \frac{\Delta\left(\frac{d\psi}{dn}\right)}{\Delta n} = \frac{d^2\psi}{dn^2}.$$

Equation (1.38) reduces to the differential equation

$$\frac{d^2\psi}{dn^2} + m^2\psi = 0. \qquad (1.39)$$

Let us examine the equation in finite differences (1.38). It it is possible to rewrite in the form

$$\psi_{n+1} - 2\psi_n - \psi_{n-1} - m^2\psi_n = 0.$$

Let us substitute in this equation the predicted solution in the form $\psi_n = \lambda^n$. Value λ proves to be the square root equation

$$\lambda^2 - (2 - m^2)\lambda - 1 = 0.$$

Assuming/setting

$$2 - m^2 = 2\cos\mu,$$

we obtain

$$\lambda = \cos\mu \pm \sqrt{\cos^2\mu - 1} = e^{\pm i\mu}.$$

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In the general case the solution of equation (1.38) takes the form (arbitrary constants are omitted):

$$\psi_n = e^{\pm in\mu}.$$

Depending on value m^2 the parameter μ can be real or complex.

Thus, the solution of difference equation differs from the solution of differential equation in terms of the frequency

$$m = 2\sin\frac{\mu}{2}.$$

Frequencies of both solutions are approximately equal to (μ~~m) with m<<1. In the course of time the solutions diverge; they remain close ones to each other on a time interval which is the larger, the less the parameter m. Hence it is apparent that the condition of replacing the difference equation to the differential equation coincides with the condition under which it is possible to disregard the corrections of the second approximation/approach to a factor of transit time and to length of the period of the structure: specific acceleration must be little in comparison with the given particle speed. Let at the length of linear accelerator be placed $n_{макс}$ the periods of the accelerating structure. Then toward the end of accelerator both solutions are radiated on the phase to value $\Delta\Psi = (\mu - m)\, n_{макс}$. After decomposing m in the power series on μ and after being restricted to the first nonvanishing approximation/approach, we will obtain

$$\frac{\Delta\Psi}{2\pi} \approx \frac{m^3}{48\pi} n_{макс}. \qquad (1.40)$$

Let us examine a numerical example. In some proton linear accelerators (I-100, CERN, Brookhaven, etc.) it is accepted $W_\lambda =$ $=2.7\cdot10^{-3}$. Assuming/setting $\cos\varphi_s = 0.8;\ \beta \approx 0.2;\ k = 1,$ we have $\frac{\Delta\Psi}{2\pi} = 10^{-4} n_{макс}$. Hence it is apparent that even at very high values $n_{макс}$, on the order of hundred, difference in both solutions in the ion accelerators can be disregarded/neglected. Virtually matter is still happier, since at fixed value ψ_λ parameter m decreases with an increase in the energy of particles. In the electron accelerators specific acceleration usually proves to be very large in view of the low value of rest energy of electrons. Therefore the replacement of difference equations to the differential equations for the electrons is virtually unacceptable.

Let us note that the solution of differential equation (1.39) is always oscillator, if only $\varphi_s < 0$. However, the solution of difference equation at the high values of m can prove to be aperiodic.

Thus, the replacement of difference equations to the differential equations is possible, if a number of periods of high-frequency structure, which fall for one period of longitudinal vibrations $N=2\pi/\mu$, much more than unity; in other words, if a phase difference ψ little is changed in one period of structure.

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Upon transfer from the finite increments to the derivatives we replace the discrete/digital effect of the field of standing waves to the particle by continuous effect. This continuous effect is reduced to the particle acceleration in the equivalent traveling wave. The sufficiently complete differential equations of longitudinal vibrations taking into account variable speed of synchronous particle and relativistic relationships/ratios it is possible to obtain on an example of equation (1.39), compiling an equation in the finite differences and replacing further the final ones of increase by derivatives. However, it is expedient to introduce into the examination the equivalent traveling wave, which reduces to the same differential equations, but by simpler and more graphic method. The approximation of standing waves by the equivalent traveling wave is valid under the same assumption about smallness $W_\lambda \beta$, as transition from the difference equations to the differential equations.

A change of the energy of arbitrary particle at any point of field is determined by the equation

$$\frac{dW}{dz} = eE_z(z, t). \qquad (1.41)$$

In expression (1.41) is omitted the dependence of the longitudinal component of field on the transverse coordinates. Here and subsequently the longitudinal vibrations of particles will be examined under the simplifying assumption that the amplitude of the longitudinal component of field is constant in entire section of aperture. This proposition it is made sufficiently well, if a radius of aperture is small in comparison with value $\beta\lambda$, since the dependence of longitudinal component on ratio $r/\beta\lambda$ in the systems with the axial symmetry is assigned by the modified Bessel function of zero-order, that has very flat extremum in zero. Function $E_z(z, t)$

is assigned by equality (1.4). Amplitude distribution of the longitudinal component of standing wave can be represented in this period of structure by Fourier series. As it was shown above, with an accuracy to the corrections of the second approximation/approach, in distribution $E_g(z)$ it suffices to consider in the dependence on the construction/design of the accelerating element/cell only even or odd part. We will be restricted to the case when function $E_g(z)$ is approximated by its even part (relative to the electrical center of period)

$$E_g(z)=\sum_{m=0}^{\infty} A_m \cos\frac{2\pi m}{L} z.$$

The instantaneous value/significance of the longitudinal component of field takes the form

$$E_z(z,\ t)=\sum_{m=-\infty}^{m=\infty} B_m \cos\left(\omega t-\frac{2\pi m}{L} z\right), \qquad (1.42)$$

where $B_0=A_0$; $B_m=\frac{1}{2}A_m$ with $m\neq 0$.

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Series/row (1.42) represents the sum of the running harmonics, which are spread in the positive (m>0) and negative (m<0) directions. Substituting series/row (1.42) into expression (1.5) and assuming/setting in the first approximation, $\omega t(z)=(\omega z/v)+\varphi$, for a partial increase in energy of arbitrary particle in the period we obtain the expression

$$\Delta W = e \sum_{m=-\infty}^{\infty} B_m \int_L \cos\left[\frac{2\pi}{L}(k-m)z - \varphi\right] dz.$$

Since

$$\int_L \cos\left[\frac{2\pi}{L}(k-m)z - \varphi\right] dz = \begin{cases} L\cos\varphi & \text{при}^{(1)} \ m = k, \\ 0 & \text{при}^{(1)} \ m \neq k. \end{cases}$$

Key: (1). with.

then

$$\Delta W = eB_k L \cos\varphi.$$

Thus, in the first approximation, the contribution in the acceleration gives only the one running harmonic of series/row (1.42), which corresponds m=k and spreading in the direction of particle motion. We will be restricted in series/row (1.42) to harmonic m=k and will consider that the speed of synchronous particle, which is determining value L, is variable/alternating along the axis of accelerator. We will obtain

$$E_z(z,\ t) = B_k \cos\omega\left(t - \int_0^z \frac{dz}{v_s}\right).$$

Further, according to expression (1.14) $B_k = \frac{1}{2}$; $A_k = E_0 T$. As a result we obtain the following expression for the equivalent traveling wave

$$E_z(z,\ t) = E_0 T \cos\omega\left(t - \int_0^z \frac{dz}{v_s}\right). \qquad (1.43)$$

Let us form an equation of the longitudinal vibrations of particles in the traveling wave. For the arbitrary particle

$$\frac{dW}{dz} = eE\cos\varphi, \qquad (1.44)$$

where φ - instantaneous phase of traveling wave at the point in which is located the particle at moment/torque t,

$$\varphi = \omega\left(t - \int_0^z \frac{dz}{v_s}\right); \qquad (1.45)$$

where E - amplitude of the traveling wave.

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In the field of standing waves $E=E_0T$. For the synchronous particle

$$\frac{dW_s}{dz} = eE\cos\varphi_s. \qquad (1.46)$$

Let us select as the dynamic variable/alternating a phase difference of the nonsynchronous and synchronous particles

$$\psi = \varphi - \varphi_s \qquad (1.47)$$

and a reciprocal difference in their energies

$$p_\psi = W_s - W. \qquad (1.48)$$

Further calculations let us conduct in the approximation/approach, with which it is unimportant, are taken these differences at one point of space, or in a moment of time. Deducting equation (1.44) from equation (1.46) and converting/transferring from the differentiation with respect to the coordinate to the differentiation with respect to time, we obtain

$$\frac{dp_\psi}{dt} = eEv_s[\cos\varphi_s - \cos(\psi + \varphi_s)]. \qquad (1.49)$$

We differentiate equality (1.45):

$$\frac{d\psi}{dt} = -\omega \frac{v - v_s}{v_s}. \qquad (1.45a)$$

From the general/common/total relativistic relationships/ratios $p = m_0\gamma v$, $\frac{dW}{dp} = v$ it follows

$$\frac{\Delta v}{v_s} \approx \frac{1}{\gamma^2} \cdot \frac{\Delta p}{p_s}; \ \Delta W \approx v_s \Delta p.$$

or

$$\frac{v - v_s}{v_s} \approx -\frac{1}{\gamma^2} \cdot \frac{p_\psi}{p_s v_s}; \ \frac{p - p_s}{p_s} \approx -\frac{p_\psi}{p_s v_s}. \qquad (1.50)$$

Hence

$$\frac{d\psi}{dt} = \frac{\omega}{\gamma^2 p_s v_s} p_\psi. \qquad (1.51)$$

Equations (1.49), (1.51) are the system of two.first-order equations, which describes the longitudinal vibrations of particles in the equivalent traveling wave. These equations can be written in the form

$$\frac{d\psi}{dt} = \frac{\partial H}{\partial p_\psi}; \ \frac{dp_\psi}{dt} = -\frac{\partial H}{\partial \psi}, \qquad (1.52)$$

where

$$H(\psi, p_\psi, t) = \frac{\omega}{2\gamma^2 p_s v_s} p_\psi^2 + eEv_s[\sin(\psi + \varphi_s) - \psi \cos\varphi_s]. \qquad (1.53)$$

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As is known, the equations of motion, written in the form (1.52), are called canonical, and the variable/alternating, in which is possible this form of writing of equations of motion - canonical-conjugated/combined. Function H from canonical-conjugated/combined variable/alternating and time - Hamiltonian of particle. Thus, variable/alternating p_ψ is the generalized momentum, canonical-conjugated/combined with the generalized coordinate of motion ψ. The special importance of the examination of motion in canonical-conjugated/combined variable/alternating is connected with existence two fundamental of the theorems of mechanics. It is expedient to briefly resemble content and value/significance of these theorems. It is in detail to the Hamiltonian mechanics it is possible to be introduced, for example, according to book [21].

During the analysis of motion of particles it is convenient to utilize a concept of phase space. Let the particle have n degrees of freedom. In the general case of motion in the three-dimensional space a number of degrees of freedom is equal to three. The state of particle at any moment of time is determined by the values 2n of the variable/alternating from which n of variable/alternating determine the position of particle in the space (generalized coordinates $q_1, \ldots, q_n$). and other n variables characterize the speed of motion by each degree of freedom (generalized momenta $p_1, \ldots, p_n$). At each moment of the time of the state of the given particle it is possible to determine by the position of the representative point in 2 n-dimensional coordinate systems. Along the axes of coordinate system plot/deposit the values of generalized coordinates and particle momenta at the given instant. This 2n-dimensional coordinate system is called phase space. To a change of the state of particle in the time in accordance with the equations of motion corresponds the displacement of the representative point in the phase space. The path of the representative point is called phase particle trajectory (see [22-24]). To periodic processes correspond locked phase trajectories. Most clearly appears phase space for the single degree-of-freedom particle, i.e., for the particle motion by which is described by two differential first-order equations. In this case phase space degenerates into the phase plane. As it is easy to show, to harmonic motion corresponds phase-plane ellipse. If we have a beam of particles, then the representative points of beam can fill in the phase space certain different from zero volumes, called the phase volume of beam. The presence of the final phase volume of beam is a consequence of the statistical scattering of initial conditions, connected usually with the disordered thermal particle motion. A number of representative points in the unit of phase volume is called phase particle density. Let in the element/cell of the phase volume

$$dV = dq_1, \ \ldots . \ dq_n \, dp_1, \ \ldots . \ dp_n$$

be located dN the representative points; phase particle density is equal to

$$n(q_1, \ \ldots, \ q_n, \ p_1, \ \ldots, \ p_n, \ t) = \frac{dN}{dV}.$$

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If particle motion is described by system of 2n the first-order equations, then in the general case there is 2n-1 constants of motion. The interval of motion is the function of dynamic variable/alternating and time, that remains constant with the particle motion in accordance with the assigned equations, in other words, in the remaining constant value in this phase trajectory.

The first of the theorems mentioned above asserts that the Hamiltonian, which does not depend clearly on the time, is constant of motion:

$$\frac{dH}{dt}=\frac{\partial H}{\partial t}.$$

The Hamiltonian of particle - this is total energy of motion (kinetic plus potential), expressed through canonical-conjugated/combined variable/alternating. Therefore this theorem essentially expresses the law of conservation of energy in the isolated/insulated system. The dynamic system whose Hamiltonian does not depend clearly on time, is called conservative.

The second theorem (Liouville's theorem): the phase volume of the collective of the particles, which move in accordance with the equations of motion, in the space of canonical-conjugated/combined variable/alternating it is invariant value. The phase volume of beam can be deformed, but by any force fields cannot glow to zero phase volume of finite quantity. It is important to note that in the case when the vector potential of electromagnetic field is equal to zero at all points of configuration space, the Cartesian coordinates x, y, z and impulses/momenta/pulses $p_x = m\dot{x}$, $p_y = m\dot{y}$, $p_z = m\dot{z}$

canonical-conjugated/combined. As the corollary of Liouville, any decrease of the scatter of particles on the coordinates (compression of beam) is accompanied by an increase in the scatter in the impulses/momenta/pulses, and vice versa, the expansion of beam causes the decrease of the scatter of particles on the impulses/momenta/pulses.

Let us examine certain limited phase volume around the representative point, which moves along the phase trajectory. Initial conditions uniquely determine phase trajectory, so that the phase trajectories of different particles at the regular points of phase space do not intersect. Hence it follows that not one representative point can cross the boundary of the chosen phase volume. A number of representative points in the element/cell of phase volume is kept constant. From Liouville's theorem it follows that the phase particle density in the space of canonical-conjugated/combined variable/alternating is constant along the phase trajectory.

Canonical-conjugated/combined variable/alternating it is possible to introduce only for the dynamic systems, in which are absent dispersive forces. Therefore Liouville's theorem is valid only in the absence of dispersive forces. In the general case dispersive forces are called the forces, which depend on the projection of particle speed on line of force [77]. These forces appear, in particular, in the presence of friction, energy losses to the radiation/emission or during the collisions of particles.

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Dispersive forces are absent in the potential fields where the force not at all depends on speed. The Lorentz force, velocity-dependent of particles, also is not dispersive, since the direction of the action of this force is perpendicular to speed. In the linear accelerators usually it is possible to disregard all sources of dispersive forces.

Let us return to the equations of longitudinal vibrations. The Hamiltonian of particle (1.53) depends clearly on time, since parameters p_s, v_s are the assigned functions of time. However, let us assume that parameters p_s, v_s are changed sufficiently slowly, so that for the time of a substantial change in variable/alternating ψ, p_ψ these parameters remain almost constant. This change in the parameters is called adiabatic. With an adiabatic change in the parameters is changed the type of phase trajectories, but it is possible to show [21] that the area, included on the plane by the locked phase trajectory, in this case is not changed. The values, which remain constants with an adiabatic change in the parameters, are called adiabatic invariants. The area, included by the locked phase trajectory, is one of the adiabatic invariants. The invariance of area - this is one of the corollaries of Liouville, since the locked trajectory can be represented as the boundary of the phase volume of certain collective of single degree-of-freedom particles.

A question about the capture of particles into acceleration mode can be examined in conservative approximation/approach $\frac{\partial H}{\partial t} = 0$ [25]. Subsequently let us show that the explicit dependence of Hamiltonian on the time leads to damping of longitudinal vibrations. Therefore calculation of capture in the conservative approximation/approach knowingly gives supply according to a number of seized particles. Second term of Hamiltonian (1.53)

$$V(\psi) = eEv_s\,[\sin(\psi + \varphi_s) - \psi\cos\varphi_s] \qquad (1.54)$$

is the analog of potential energy. In the conservative approximation/approach the Hamiltonian is integral of motion; therefore equation (1.53) is the equation of phase trajectory. To each phase trajectory corresponds the specific value/significance H:

$$p_\psi = \pm\gamma\sqrt{\frac{2p_sv_s}{\omega}}\sqrt{H - V(\psi)}. \qquad (1.55)$$

Expression (1.55) gives the simple method of the construction of the family of phase trajectories [23, 24]. Let us construct on plane ψ, V potential function $V(\psi)$ (Fig. 1.5). The relief of potential function is determined by the sum of sinusoid $\sin(\psi + \varphi_s)$, of out of phase relative to zero, and straight line $-\psi\cos\varphi_s$. Let us conduct on plane ψ, V horizontal line V=H. Actual values p_ψ occur only when $V(\psi) < H$ and for each value/significance ψ they are determined by root of the current difference H-V.

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To value/significance H_0, at which horizontal line $V=H_0$ concerns potential function at the point of the minimum, corresponds the isolated/insulated singular point on plane ψ, p_ψ. This singular point is called center. In the nearest vicinity of center phase trajectories are locked. The direction of the motion of the representative points (see Fig. 1.5) is obtained from the condition: $\frac{d\psi}{dt}>0$ when $p_\psi>0$ (1.51). With increase in H the framework of the locked phase trajectories grows/rises. Let when $H=H_c$ the horizontal line concern curve $V(\psi)$ at the point of maximum. To this value/significance of Hamiltonian corresponds the special phase trajectory, which has the point of self-intersection on the axis of abscissas. The coordinate of the point of self-intersection is an abscissa of maximum $V(\psi)$; the indicated point is also the singular point of equations of motion (saddle). The phase trajectory, passing through the saddle, is called separatrix. Separatrix divides two regions of phase plane with different character of phase trajectories. Within the separatrix phase trajectories are locked, out of the separatrix extended.

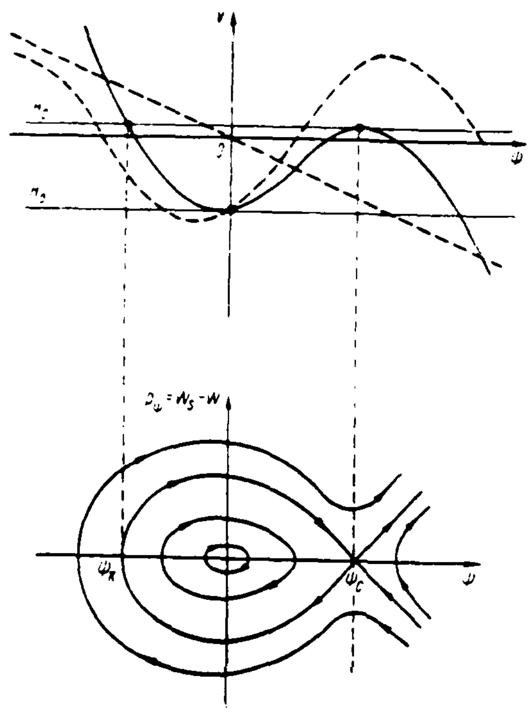

Fig. 1.5.

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Let us note that as a whole the family of phase trajectories on plane ψ, p_ψ corresponds to the possible motions of the ball/sphere, which wheels without friction along the relief, assigned by potential function. Center $\psi = 0, p_\psi = 0$ corresponds to the synchronous particle: $\varphi = \varphi_s = \text{const}; W = W_s(z)$. Phase and energy of the particles, which caught upon the injection inside the separatrix, complete oscillations/vibrations around the synchronous phase and the current energy of synchronous particle. These particles are accelerated. Energy of the particles, which proved to be out of the separatrix, decreases relatively $W_{(s)}$. It is obvious that such particles are not seized into acceleration mode. Thus in the conservative approximation/approach separatrix limits the capture region of particles into acceleration mode. Thus in the conservative approximation/approach separatrix limits the capture region of particles into acceleration mode.

The coordinates of the extrema of potential function are determined by the condition

$$\frac{dV}{d\varphi}=eEv_s\left[\cos\left(\psi-\varphi_s\right)-\cos\varphi_s\right]=0. \quad (1.56)$$

Equation (1.56) in the period in question has two roots: $\psi=0$ and $\psi_c=-2\varphi_s$. The second derivative at the points of extrema is equal to

$$\frac{d^2V}{d\psi^2}(0)=-eEv_s\sin\varphi_s; \quad \frac{d^2V}{d\psi^2}(-2\varphi_s)=eEv_s\sin\varphi_s.$$

The chosen value/significance of synchronous phase is stable, if the second derivative in zero is positive. To this corresponds condition $\sin\varphi_s<0$. Then the second state of equilibrium $\psi_c=-2\varphi_s$ (or $\varphi=-\varphi_s$) is unstably and saddle.

According to expression (1.55), the separatrix is described by the equation

$$p_\varphi=\pm\gamma\sqrt{\frac{2p_sv_s}{\omega}}\sqrt{V(-2\varphi_s)-V(\psi)}. \quad (1.57)$$

In the practical calculations of basic interest is a relative difference in the impulses/momenta/pulses

$$g = \frac{p - p_s}{p_s}. \tag{1.58}$$

Utilizing relationship/ratio (1.50) and expression (1.54), we obtain the equation of separatrix in coordinates ψ, g:

$$g_c(\psi) = \pm \gamma \sqrt{\frac{2eE}{\omega p_s}} \sqrt{(\psi - \sin\psi + 2\varphi_s)\cos\varphi_s - (1 - \cos\psi)\sin\varphi_s}. \tag{1.59}$$

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The semirange of separatrix along the axis g is equal to

$$g_{\text{макс}} = 2\gamma \sqrt{\frac{eE}{\omega p_s}} \sqrt{\varphi_s \cos\varphi_s - \sin\varphi_s}. \tag{1.60}$$

Coordinate ψ_K of the second point of intersection of separatrix with the axis of abscissas let us find from the equation

$$V(\psi_K) = V(-2\varphi_s).$$

Substituting into this equation expression (1.54) and expanding trigonometric functions from φ_s, ψ in the power series, we obtain

$$\psi_K \approx \varphi_s - \frac{1}{10}\varphi_s^3 - \ldots \tag{1.61}$$

With an accuracy sufficient for practical purposes it is possible to accept $\psi_K = \varphi_s$. Fig. 1.6 gives separatrix in coordinates ψ, g.

With expressions (1.59), (1.60) more conveniently to operate, after introducing in them the value of specific acceleration W_λ. Substituting expression (1.25a) into equalities (1.59), (1.60), we have

$$g_c(\psi) = \pm\sqrt{2\gamma}\,\frac{\Omega_c}{\omega}\times$$
$$\times\sqrt{1-\cos\psi-(\psi-\sin\psi-2\varphi_s)\,\mathrm{ctg}\,\varphi_s};$$
(1.59a)

$$g_{\text{макс}} = 2\sqrt{\gamma}\,\frac{\Omega_c}{\omega}\sqrt{1-\frac{\varphi_s}{\mathrm{tg}\,\varphi_s}}, \quad (1.60a)$$

where

$$\Omega_c = \omega\sqrt{\frac{W_\lambda\ \mathrm{tg}\,\varphi_s}{2\pi\beta_s}}. \quad (1.62)$$

Let us tentatively rate/estimate value $g_{\text{макс}}$. Let $W_\lambda = 2.7\cdot10^{-3}$; $\cos\varphi_s = 0.8$; $\beta_s = 0.038$ (it corresponds to energy of injection W_0=700 keV). Then $\frac{\Omega_c}{\omega} = 0.094$ and $g_{\text{макс}} = 7.3\%$.

If upon the injection it is possible to completely fill capture region (see Fig. 1.6), then clusters subsequently do not fluctuate; the longitudinal length of clusters monotonically increases according to the law

$$\Delta z = 3\,\varphi_s\,\frac{v_s}{\omega}.$$

But if into the accelerator is injected monochromatic beam with the scatter of impulses/momenta/pulses, substantially smaller than the spread/scope of separatrix (Fig. 1.7), then clusters upon the acceleration fluctuate along the length: the region, occupied with the representative points of beam, rotates within the separatrix with the frequency of longitudinal vibrations. The equations of longitudinal vibrations are nonlinear. Therefore frequency depends on the amplitude: with an increase in the amplitude the frequency of longitudinal vibrations decreases.

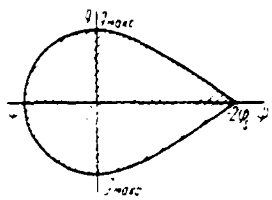

Fig. 1.6.

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This causes the distortion of phase volume, schematically shown in Fig. 1.7.

The calculation of capture region taking into account the explicit dependence of Hamiltonian on the time can be carried out numerically [26]. The qualitative picture of the distribution of phase trajectories for this case is shown in Fig. 1.8. As already mentioned, when $\frac{\partial H}{\partial t}=0$ the phase oscillations attenuate. Therefore the family of phase trajectories in Fig. 1.8 corresponds to the possible motions of the ball/sphere, which wheels along the potential relief, but in the presence of friction. Capture region in Fig. 1.8 is shaded. It proves to be open-circuited. Calculation shows that in the virtually acceptable initial parameters of ionic accelerator $\left(\frac{\Omega_c}{\omega}\sim 0{,}1\right)$ the capture region on the phases at the level g=0 proves to be approximately/exemplarily to 10o/o wider than the region, designed in the conservative approximation/approach. The maximum width of capture region on the phases occurs with Δg=+2-4o/o and is approximately/exemplarily to 20-30o/o more than 3 q. These differences in the conservative and dissipative approximations/approaches can have certain value/significance upon the injection of monochromatic beam.

For evaluating the accuracy of differential equations Fig. 1.9 gives the separatrix of difference equation.

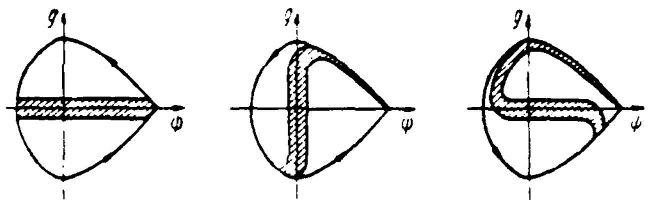

Fig. 1.7.

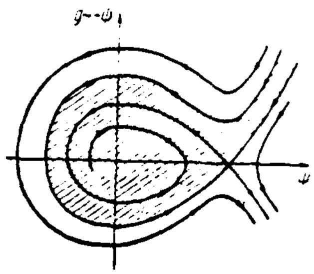

Fig. 1.8.

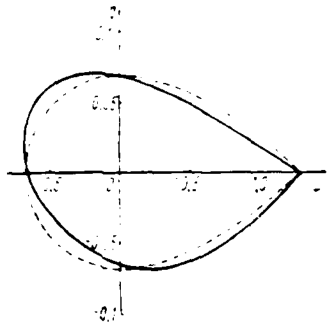

Fig. 1.9.

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Separatrix is obtained from the difference equations of numerical calculation in the conservative approximation/approach when $\cos\varphi_0 = 0{,}8$; $\Omega_c/\dot{\omega} = 0.094$. Phase ψ — in the clearance, which follows after drift tube where is measured value/significance g. In Fig. 1.9 where the comparison dotted line showed the separatrix, obtained from the differential equations at the same values of the parameters. Difference in both separatrices is virtually unessential.

Let us examine the now limiting case of particle acceleration in the traveling wave with the phase speed, equal to the speed of light. In this wave synchronous particle is absent, since for any material particle with a finite mass of rest of v<c. Rate of change in the impulse/momentum/pulse of any particle is equal to

$$\frac{dp}{dt} = eE\cos\varphi.$$

where φ - instantaneous phase of wave at the point in which is located the particle at the moment of time t

$$\varphi = \omega t - \frac{\omega z}{c}. \qquad (1.63)$$

Let us introduce for the simplification the given particle momentum

$$p_\varphi = \frac{p}{m_0 c}$$

and the specific particle acceleration in the antinode of the traveling wave, determined by the equality

$$W_\lambda = \frac{eE\lambda}{2\pi\mathcal{E}_0}. \qquad (1.64)$$

This value differs from W_λ upon the acceleration in the wave where $v_\Phi < c$ [see equation (1.25)]. Differentiating expression (1.63) and taking into account that

$$\frac{v}{c} = \frac{p_\varphi}{\sqrt{1+p_\varphi^2}},$$

we obtain the following equations of motion of particle in the field of the traveling wave with a phase speed of $v_\Phi = c$:

$$\frac{d\varphi}{dt} = \omega\left[1 - \frac{p_\varphi}{\sqrt{1+p_\varphi^2}}\right];$$

$$\frac{dp_\varphi}{dt} = \omega W_\lambda \cos\varphi. \qquad (1.65)$$

Hence

$$\frac{dp_\varphi}{d\varphi} = W_\lambda \frac{\cos\varphi}{1 - \frac{p_\varphi}{\sqrt{1+p_\varphi^2}}}. \qquad (1.66)$$

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Integrating expression (1.66), we obtain the family of the phase trajectories

$$\sqrt{p_\varphi^2+1}-p_\varphi=C-W_\lambda\sin\varphi,$$

where C - arbitrary integration constant. The equation of the family of phase trajectories is convenient to represent in the parametric form, after introducing auxiliary variable/alternating

$$\xi(p_\varphi)=\sqrt{p_\varphi^2+1}-p_\varphi,$$

positive at all actual values of impulse/momentum/pulse. Then

$$p_\varphi=\frac{1-\xi^2}{2\xi};\qquad (1.67)$$
$$\xi=C-W_\lambda\sin\varphi.$$

The dependence of the parameter ξ on the phase φ is given in Fig. 1.10. For curves 1, 2 (see Fig. 1.10) $C>W_\lambda$; for curved 3 $C=W_\lambda$; for curved 4, 5 $C<W_\lambda$. It is easy to see that to passage to the limit $p_\varphi\to\infty$ it corresponds $\xi\to 0$. Consequently, into acceleration mode are seized particles with the initial conditions under which $C\leqslant W_\lambda$. When $C>W_\lambda$ the particle momentum always remains finite quantity. The boundary of the region of capture corresponds $C=W_\lambda$ or $\xi=W_\lambda(1-\sin\varphi)$. Phase trajectories on plane φ, p_φ are given in Fig. 1.11. The direction of the motion of the representative points is selected from condition $\frac{dp_\varphi}{dt}>0$ with cosφ>0, according to equation (1.65). The boundary of the region of capture is isolated with heavy line. The maximum values of impulses/momenta/pulses for the particles, not seized into acceleration mode, fall to the points $\varphi=\pi/2\pm 2\pi n$. The impulses/momenta/pulses of these particles periodically are changed about the constant value, and particles monotonically lag on the phase behind the traveling wave, converting/transferring from one period to another.

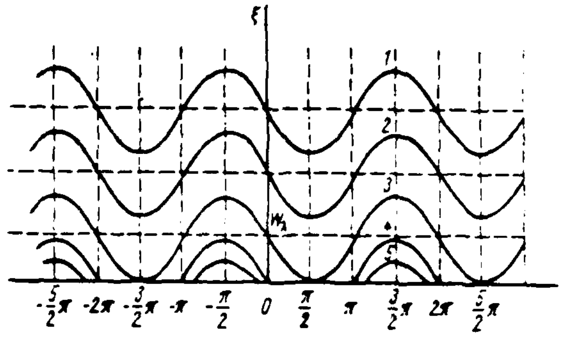

Fig. 1.10.

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The phase trajectories, which lie higher than boundary curve, correspond to the particles, seized into acceleration mode. In the course of time the speed of the seized particles always begins to grow/rise. Each particle is seized by the specific period of the traveling wave; particle remains from the wave, but its phase, according to expression (1.67), it asimptotically tends for the value/significance

$$\varphi_\infty = \mathrm{Arc}\sin\frac{C}{W_\lambda}.$$

Phase oscillations when $v_\Phi = c$ are absent. The minimum value/significance of impulse/momentum/pulse for the particles, which initially caught into the negative half-period of wave, falls to the point $\varphi=-\pi/2$. The clusters of the accelerated particles are collected in each period of the traveling wave in the region from $\varphi=-\pi/2$ to

$\varphi=0$. The boundary of the region of capture is described by the equation

$$p_\varphi = \frac{1 - W_\lambda^2 (1-\sin\varphi)^2}{2W_\lambda (1-\sin\varphi)}. \quad (1.68)$$

The value/significance of the given impulse/momentum/pulse minimally permissible for the capture is equal

$$p_{\varphi\,\text{мин}} = \frac{1 - 4W_\lambda^2}{4W_\lambda}. \quad (1.69)$$

Let the amplitude of the traveling wave be equal to 30 kV/cm and λ=10 cm. Then the value of specific acceleration for the electrons proves to be equal to $W_\lambda = 0.1$, and for protons $W_\lambda = 5 \cdot 10^{-5}$.

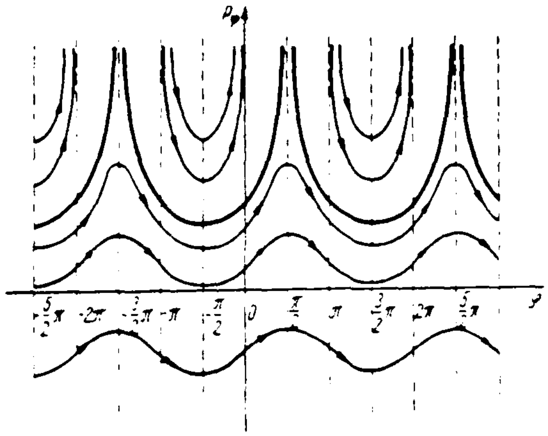

Fig. 1.11.

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Hence it is apparent that the electrons are seized by the traveling wave, which has phase speed v_ϕ c already with energies on the order of 1 MeV, and protons - only with the energies, which exceed 10^3 GeV. Thus, in the traveling wave with a phase speed of $v_\phi = c$ virtually it is possible to accelerate only electron beams.

§1.3. Small longitudinal vibrations.

Particles, close ones upon the injection to the synchronous ones, vary on the phase with a small spread/scope, their behavior can be described by the linearized equations. The investigation of the linearized equations makes it possible sufficient simply to calculate fading and phase change of longitudinal vibrations at the length of accelerator, which is important from the point of view of the estimation of energy spectrum of the accelerated particles. The estimation of certain allowances for the production of the accelerating system also is reduced to the analysis of solutions in the linear approximation/approach.

Let $\psi \ll 1$. Decomposing/expanding the right side of equation (1.49) in the series/row according to degrees ψ and being limited to linear term, we have

$$\frac{dp_\psi}{dt} = eEv_s \sin\varphi_s \psi.$$

According to expressions (1.25), (1.62),

$$-eE\sin\varphi_s = \frac{p_s}{\omega\gamma}\Omega_r^2. \qquad (1.70)$$

As a result we obtain the following system of equations of first-order, which describes the small oscillations

$$\frac{d\psi}{dt} = \frac{\omega}{\gamma^2 p_s v_s} p_\psi;$$
$$\frac{dp_\psi}{dt} = -\frac{\omega p_s v_s}{\gamma}\left(\frac{\Omega_r}{\omega}\right)^2 \psi. \qquad (1.71)$$

System (1.71) is reduced to one equation of the second order

$$\frac{d^2\psi}{dt^2} + \frac{d}{dt}(\ln \gamma^2 p_s v_s)\frac{d\psi}{dt} + \gamma^{-3}\Omega_r^2\psi = 0. \qquad (1.72)$$

The coefficient of equation (1.72) they depend on the current energy of synchronous particle and, according to the condition, they are the "slow" functions of time. Let us switch over to the dimensionless independent variable, which will make it possible to conduct a comparative evaluation of coefficients. Let us assume $\tau=\omega t$. Then

$$\frac{d^2\psi}{d\tau^2} - 2\delta(\tau)\frac{d\psi}{d\tau} - |\nu(\tau)|^2\psi = 0. \qquad (1.73)$$

where

$$\delta(\tau) = \frac{1}{2\omega}\cdot\frac{d}{dt}(\ln\gamma^2 p_s\omega) \approx \frac{3\gamma^2-1}{2\gamma^3}\left(\frac{\Omega_c}{\omega}\right)^2 \mathrm{ctg}\,\varphi_s, \qquad (1.74)$$

$$\nu(\tau) = \gamma^{-\frac{3}{2}}\frac{\Omega_c}{\omega} \qquad (1.75)$$

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Let us note that when $W'_\lambda = \mathrm{const}$

$$\delta(\tau) = \frac{3\gamma^2-1}{2\gamma^2}\cdot\frac{1}{\tau}; \quad \nu^2(\tau) = \frac{1}{\gamma^2}\cdot\frac{|\mathrm{tg}\,\varphi_s|}{\tau}. \qquad (1.76)$$

Actually/really, from relationship/ratio (1.26) with an accuracy to the replacement of finite increments by derivatives we have

$$\frac{1}{m_0c}\cdot\frac{dp_s}{dt} \approx \frac{\omega}{2\pi}W'_\lambda.$$

Hence

$$\tau \approx 2\pi\gamma\frac{\beta_s}{W'_\lambda}, \qquad (1.77)$$

or, taking into account expression (1.62),

$$\frac{1}{\tau} \approx \frac{1}{\gamma}\left(\frac{\Omega_c}{\omega}\right)^2|\mathrm{ctg}\,\varphi_s|.$$

Substituting latter/last expression into equality (1.74), (1.75), we obtain equalities (1.76). Functions $\delta(\tau)$, $\nu^2(\tau)$ are hyperbolas; they are slow functions τ with τ>>1. Thus, the smallness of relation $\frac{W'_\lambda}{\beta_s}$ guarantees the slowness of the coefficients of equation (1.73).

Let us represent the solution of equation (1.73) in the form

$$\psi(\tau) = \Phi(\tau)\sin\Psi(\tau). \qquad (1.78)$$

Let us now substitute solution (1.78) in equation (1.73) and will gather coefficients when $\sin\Psi$ and $\cos\Psi$:

$$\frac{d^2\Phi}{d\tau^2}-2\delta\frac{d\Phi}{d\tau}-\Phi\left[\nu^2-\left(\frac{d\Psi}{d\tau}\right)^2\right]=0;$$
$$2\frac{d\Psi}{d\tau}\left(\frac{d\Phi}{d\tau}-\delta\Phi\right)+\Phi\frac{d^2\Psi}{d\tau^2}=0. \qquad (1.79)$$

Due to the "slowness" of the coefficients of equation (1.73) we have $\Phi\sim e^{-\delta}\sim 1$. Since $\delta\sim 1/\tau$, then $\dot\Phi\sim\frac{1}{\tau^2}$, $\ddot\Phi\sim\frac{1}{\tau^3}$. Disregarding in first equation (1.79) the members of order $1/\tau^3$ in comparison with $\nu^2\sim 1/\tau$, we obtain

$$\frac{d\Psi}{d\tau}=\nu(\tau). \qquad (1.80)$$

Finally, disregarding in the second equation term $\dot\Psi\dot\Phi\sim\tau^{-5/2}$ in comparison with terms $\delta\dot\Psi$, $\ddot\Psi\sim\tau^{-3/2}$, we obtain the equation

$$\frac{1}{\Phi}\cdot\frac{d\Phi}{d\tau}=-\delta(\tau)-\frac{1}{2\nu(\tau)}\cdot\frac{d\nu}{d\tau}. \qquad (1.81)$$

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From equalities (1.75), (1.80) it follows

$$\frac{d\Psi}{dt}=\Omega_c\gamma^{-\frac{3}{2}}.$$

Thus, value

$$\Omega=\gamma^{-3/2}\Omega_c \qquad (1.82)$$

is an instantaneous value/significance of the frequency of small longitudinal vibrations. Parameter Ω_c, determined by expression (1.62), is nonrelativistic approximation/approach to frequency of small longitudinal vibrations. Phase change of oscillations at the length of accelerator is equal to

$$\Delta\Psi(t)=\int_{t_0}^{t}\Omega(t)\,dt. \qquad (1.83)$$

Fading the amplitude of phase oscillations for different functions $W_\lambda(z)$ can be obtained by the integration of equation (1.81). However, to more simply determine fading, on the basis of the adiabatic invariant. Let us limit ourselves in potential function (1.54) to the first nonvanishing approximation/approach of ψ.

$$V(\psi) = -eEv_s \sin\varphi_s \left(\frac{1}{2}\psi^2 - 1\right). \qquad (1.84)$$

Substituting expression (1.84) in the equation of phase trajectory (1.55) and taking into account equality (1.70), we obtain the phase trajectory of small longitudinal vibrations in the form

$$\frac{\psi^2}{\Phi^2} + \frac{p_\psi^2}{P_\psi^2} = 1, \qquad (1.85)$$

moreover

$$P_\psi = \gamma^2 p_s v_s \frac{\Omega}{\omega}\Phi.$$

Phase trajectory (1.85)is ellipse, which will agree with the harmonic character of solution (1.78). The area, included by this ellipse,

$$I_s = \pi\Phi P_\psi,$$

is constant in accordance with the theorem about the adiabatic invariant. Hence

$$\Phi = \frac{\text{const}}{\sqrt{\gamma^2 p_s v_s \frac{\Omega}{\omega}}}; \qquad (1.86)$$

$$P_\psi = \text{const}\sqrt{\gamma^2 p_s v_s \frac{\Omega}{\omega}}.$$

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The amplitude of the oscillations of a relative difference in the impulses/momenta/pulses

$$g_a = \left(\frac{p-p_s}{p_s}\right)_{\text{макс}},$$

according to expression (1.50), is equal to

$$g_a = \frac{P_\psi}{p_s v_s} \qquad (1.87)$$

and it is changed with an increase in the energy of particles according to the law

$$g_a = \text{const}\sqrt{\frac{\gamma^2}{p_s v_s}\cdot\frac{\Omega}{\omega}}. \qquad (1.88)$$

First let us examine damping small oscillations in the general relativistic case upon the uniform acceleration along axis $(W_\lambda = \text{const})$. According to expressions (1.62), (1.82), the instantaneous values of frequencies decrease with an increase in the energy as follows

$$\frac{\Omega_c}{\omega} \doteqdot \gamma^{1/2} p_s^{-1/2}; \quad \frac{\Omega}{\omega} \doteqdot \gamma^{-1} p_s^{-1/2}. \qquad (1.89)$$

Since $p_s v_s \doteqdot \gamma^{-1} p_s^2$, that

$$\Phi \doteqdot p_s^{-3/4}; \quad P_\psi \doteqdot p_s^{3/4}; \quad g_a \doteqdot \gamma p_s^{-5/4}. \qquad (1.90)$$

The amplitude of small oscillations of particles along longitudinal axis $z - z_s = \psi\frac{v_s}{\omega}$ and absolute difference in impulses/momenta/pulses $p - p_s$ are proportional to values

$$z - z_s \doteqdot \gamma^{-1} p_s^{1/4}; \quad p - p_s \doteqdot \gamma p_s^{-1/4}. \qquad (1.91)$$

From relationships/ratios (1.90), (1.91) it is evident that the phase volume of the particles, which accomplish small oscillations, in coordinates ψ, P_ψ and $z - z_s$, $p - p_s$ is retained. These pairs of dynamic variable/alternating are canonical-conjugated/combined. Coordinates ψ, g are not canonical-conjugated/combined; phase volume on plane

ψ, g with an increase in the impulse/momentum/pulse is decreased, moreover they decrease both the spread/scope of phase oscillations and the range of oscillations of a relative difference in the impulses/momenta/pulses.

Let us note that the current spread/scope of separatrix (1.60a) decreases according to the law

$$g_{\text{макс}} \doteqdot \gamma p_s^{-1/2}. \qquad (1.92)$$

Thus, the amplitude of the oscillations of a relative difference in the impulses/momenta/pulses falls more rapidly than the spread/scope of separatrix.

For evaluating the course g_a and $p-p_s$ with the high energies of formula (1.90), (1.91) it is convenient to represent in the form

$$g_a \doteqdot \gamma^{-1/4}\beta^{-5/4}; \quad p-p_s \doteqdot \gamma^{3/4}\beta^{-1/4}. \qquad (1.93)$$

A relative difference in the impulses/momenta/pulses decreases up to $\beta=1$. But absolute difference at the high speeds is begun to grow/rise; the minimum is necessary at the value/significance $\gamma^2=4/3$.

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In the nonrelativistic approximation/approach formulas (1.89)-(1.91) are reduced to the form;

$$\frac{\Omega_c}{\omega} \doteqdot W_s^{-1/4}; \quad \Phi \doteqdot W_s^{-3/8}; \quad g_a \doteqdot W_s^{-5/8}; \qquad (1.94)$$
$$z-z_s \doteqdot W_s^{1/8}; \quad v-v_s \doteqdot W_s^{-1/8}$$

(W_s – kinetic energy of synchronous particle).

The course of the oscillations of phase depending on current time $t \div \sqrt{W_s}$ is given in Fig. 1.12. Damping longitudinal vibrations is very considerable. Thus, under the ideal conditions upon the particle acceleration from the energy of injection W_0=700 keV to the final energy W=24 MeV the amplitude of phase oscillations decreases almost 4 times, and the amplitude of the oscillations of a relative difference in the impulses/momenta/pulses - 9 times. In this case the spread/scope of separatrix along vertical axis $(g_{макс})$ decreases from the beginning toward the end of the accelerator 2.4 times.

Are of practical interest the cases when acceleration along the axis of accelerator is variable [27]:

$$\frac{dW_s}{dz} = \left(\frac{dW_s}{dz}\right)_0 \left(\frac{W_s}{W_0}\right)^m. \qquad (1.95)$$

Index 0 relates to the initial values of values. A similar accelerating system can be specially designed for assigned m, and to it will correspond the equivalent traveling wave with the monotonically changing acceleration of the wave front and the respectively changing amplitude, which ensures the retention/preservation/maintaining synchronous phase. Let us note that the accelerator with drift tubes, designed for the constant

value of middle field, has acceleration variable along the axis, if the factor of transit time is changed along the axis [see expression (1.33)].

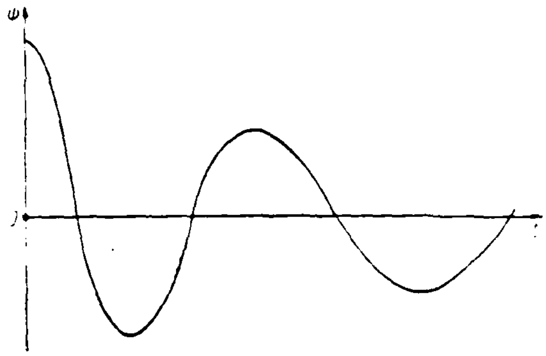

Fig. 1.12.

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With the cophasal supply of accelerating gaps, according to expression (1.95), we have

$$W_\lambda = W_\lambda^0 \left(\frac{W_s}{W_0}\right)^m .$$

But if accelerating gaps are supplied noncophasally, for example, all periods of structure have constant length $L = k\beta\lambda_{\text{эфф}} = \text{const}$, then $\lambda_{\text{эфф}} \doteqdot W_s^{-1/2}$ and $W_\lambda \doteqdot W_s^{m-1/2}$. Let us examine the case λ=const in the nonrelativistic approximation/approach. Since

$$\Phi \doteqdot \left(\frac{\Omega_c}{\omega}\right)^{-1/2} v_s^{-1},$$

then

$$\Phi \doteqdot W_s^{-\frac{3+2m}{8}}; \quad g_a \doteqdot W_s^{-\frac{5-2m}{8}} . \qquad (1.96)$$

In comparison with case of m=0 the phase oscillations with m<0 attenuate more weakly, and the oscillations of a relative difference in the impulses/momenta/pulses are more rapid. The oscillations of phase are stable, thus far m>-3/2. Therefore a maximally attainable fading of the scatter of impulses/momenta/pulses $g_a \doteqdot W_s^{-1}$. With m>0 oscillation of phase they attenuate more rapidly, but more slowly attenuate the oscillations of impulses/momenta/pulses.

Upon the injection of monochromatic beam the scatter of particles on the impulses/momenta/pulses at the output of accelerator proves to be minimum, if at the length of accelerator is placed the integer of half-waves of the longitudinal vibrations (see Fig. 1.7). Therefore the calculation of the phase change of longitudinal vibrations ψ and the explanation of its dependence on the basic parameters of accelerator for the establishment of the corresponding allowances is essential. Let us examine, first of all, the most important case when the acceleration of synchronous particle along the axis is constant. Phase change of longitudinal vibrations is determined by integral (1.83). According to expressions (1.62), (1.82),

$$\Omega = \omega \sqrt{\frac{W_\lambda \ \operatorname{tg} \varphi_s}{2\pi}} \cdot \frac{1}{\gamma \sqrt[4]{\gamma^2 - 1}},$$

where γ - Lorenz's factor for the synchronous particle. Further, differentiating equality (1.77), we obtain

$$dt = \frac{2\pi}{\omega W_\lambda} \cdot \frac{\gamma \, d\gamma}{\sqrt{\gamma^2 - 1}}.$$

In expression (1.83) let us replace the variable/alternating of integration. Then

$$\Delta\Psi = \sqrt{\frac{2\pi \, |\operatorname{tg} \varphi_s|}{W_\lambda}} \int\limits_{\gamma_0}^{\gamma} (\gamma^2 - 1)^{-1/4} \, d\gamma.$$

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Let us introduce for the convenience phase ψ, calculated off the conditional moment of time, which corresponds γ=1,

$$\Psi = \varkappa \int_{1}^{\gamma} (\gamma^2 - 1)^{-3/4} d\gamma, \quad (1.97)$$

where

$$\varkappa = \sqrt{\frac{2\pi \ \mathrm{tg}\, \varphi_s}{W_\lambda}} \quad (1.98)$$

will eat phase factor for the longitudinal vibrations. Phase change at the length of accelerator with the initial value of Lorenz's factor $\gamma = \gamma_0$ and finite value $\gamma = \gamma_K$ is equal to

$$\Delta\Psi = \Psi(\gamma_K) - \Psi(\gamma_0).$$

By the second replacement of the variable/alternating

$$x^4 = \gamma^2 - 1$$

integral (1.97) is led to tabular function [28]

$$\Psi = 2\varkappa \int_0^x \frac{dx}{\sqrt{1+x^4}} = \varkappa F\left(\varphi, \frac{1}{\sqrt{2}}\right); \quad \mathrm{tg}\, \frac{\varphi}{2} = \sqrt[4]{\gamma^2 - 1}, \quad (1.99)$$

where $F(\varphi, k)$ - the incomplete elliptical integral of the 1st kind:

$$F(\varphi, k) = \int_0^\varphi \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}.$$

The tables of function $F(\varphi, k)$ are given, for example, in book [29].

In the nonrelativistic approximation/approach $\varphi \ll 1$ and $F(\varphi, k) \approx \varphi$. Hence

$$\Delta\Psi = \varkappa(\varphi - \varphi_0).$$

For the variable/alternating φ we have

$$\varphi \approx 2\left[\left(\frac{\mathcal{E}_0 - W_s}{\mathcal{E}_0}\right)^2 - 1\right]^{1/4} \approx 2^{3/4}\left(\frac{W_s}{\mathcal{E}_0}\right)^{1/4}.$$

Thus, with the nonrelativistic ones the energy

$$\Delta\Psi = 2^{3/4}\varkappa\left[\left(\frac{W_s}{\mathcal{E}_0}\right)^{1/4} - \left(\frac{W_0}{\mathcal{E}_0}\right)^{1/4}\right]. \qquad (1.100)$$

Expression (1.100) can be also represented in the form

$$\Delta\Psi = 2\varkappa\left(\sqrt{\beta} - \sqrt{\beta_0}\right). \qquad (1.100a)$$

Let us note that the instantaneous frequency of phase oscillations (1.62), (1.82) is proportional to square root of specific acceleration, while phase factor (1.98) is inversely proportional to this root.

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Therefore with an increase in the specific acceleration the frequency of longitudinal vibrations increases, and phase change between the given values of initial and final energies decreases. This at first glance strange fact is connected with the fact that with the growth of specific acceleration decreases time of landing run of the particle between the assigned energies, so that in spite of an increase in the frequency phase change decreases.

Case of m≠0 (1.95) let us examine only with low energies. Calculation, analogous to that given, taking into account the relationships/ratios

$$\Omega^2 = \omega^2 \frac{W_\lambda^0 \operatorname{tg} \varphi_s}{2\pi\beta_0} \left(\frac{W_s}{W_0}\right)^{m-\frac{1}{2}}; \quad dt = \frac{\pi\beta_0}{\omega W_\lambda^0} \left(\frac{W_s}{W_0}\right)^{-m-\frac{1}{2}} d\left(\frac{W_s}{W_0}\right)$$

gives

$$\Delta\Psi = \frac{2^{5/4}}{1-2m} \varkappa_0 \left[\left(\frac{W_s}{\mathcal{E}_0}\right)^{1/4} \left(\frac{W_s}{W_0}\right)^{-\frac{m}{2}} - \left(\frac{W_0}{\mathcal{E}_0}\right)^{1/4}\right], \quad (1.101)$$

where $\varkappa_0$ — the initial value of phase factor (1.98). With m=0 formula (1.101) is reduced to expression (1.100).

For the orientation in the possible values of phase change of longitudinal oscillations at the length of linear accelerator let us determine $\Delta\Psi$ with the energy of injection 700 keV and exit energy of the particles of 24 MeV. Let $W_\lambda = 2{,}7 \cdot 10^{-3}$; $\cos\varphi_s = 0{,}8$. Then $\varkappa = 42$; formula (1.100a) gives for $\Delta\Psi$ the value/significance of 23.6 or 3.75 oscillations at the length of accelerator. Basic phase change occurs with the low energies. With further increase in the exit energy phase change increases slowly.

Formulas (1.99) or with low energies (1.100), (1.101) allow at the given values of energy of injection and final energy of particles to select phase factor $\varkappa$ by such so that at the length of accelerator would lie/fall/lay the integer of half-oscillations. The obtained value/significance of phase factor makes it possible to refine the value of synchronous phase and, therefore, middle field. By differentiation of expressions (1.99), (1.100) it is possible to obtain allowance for the value of middle field with assigned standard deviation $\Delta\Psi$ from computed value.

With an increase of the amplitude the frequency of longitudinal vi..ations in the general nonlinear case decreases. For the nonrelativistic speeds it is possible to obtain the following first nonlinear approximation/approach to phase change of longitudinal vibrations $(W_\lambda = \text{const})$:

$$\Delta\Psi = \Delta\Psi_{\text{л}} - \frac{1}{16}\varkappa\Phi_0^2\sqrt{\beta_0}\left(1-\frac{\beta_0}{\beta}\right), \qquad (1.102)$$

where $\Delta\psi_{\text{л}}$ — phase change of small oscillations, determined by formula (1.100a) Φ_0 — the initial amplitude of phase oscillations. It is interesting to note that with an increase in the energy of particles "nonlinear" component of phase change approaches a constant value.

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This is connected with the fact that the amplitude of longitudinal vibrations adiabatically attenuates, so that instantaneous frequency asimptotically tends for the frequency of small oscillations. For an approximate estimate let us examine the particle, initially close to the separatrix: $\varphi_0 = |\varphi_s|$. Under the conditions of preceding/previous example $(\cos\varphi_s = 0.8;\ \varkappa = 42;\ p_0$ =0.0386) we will obtain that the nonlinear correction to phase change does not exceed 0.2 rad. This value is sufficiently low.

§1.4. Longitudinal vibrations in the imperfect accelerating system.

The damage of the regular structure of the accelerating system and different instabilities of amplitude and phase of electric field bring the fluctuations of phases and particle momenta. These disturbances/perturbations can be relatively greater as, for example, the idle gaps/intervals between the resonators or an abrupt change in the specific acceleration upon transfer of one resonator to another. Any real accelerating system, even which does not contain the large interruptions/discontinuities of regularity (which are most frequently caused by design considerations and are considered during the calculation), it has relatively small errors, distributed in each period of structure according to the random law. Small errors exert a substantial influence on the characteristics of the accelerated beam in view of the recurrence of effect on the particles. In the accelerating system with drift tubes the basic sources of small random disturbances are the random errors of location of centers of the accelerating clearances along the longitudinal axis of accelerator and the nonuniformity of middle accelerating field, connected both with the random disturbances/breakdowns of geometry and with those instabilities of accelerating field in the time, which are compared in the duration with the time of flight of the particles through the accelerator. The external resonances of longitudinal vibrations in the linear accelerators usually prove to be unessential, since at the length of accelerator is placed a small number of oscillations, and frequency in this case comparatively rapidly is changed. In the accelerating systems there are systematic errors, caused by a difference in the real parameters of system from the calculated ones, for example, by an inaccuracy in the installation of the given value of accelerating field or by the instability of field from one impulse/momentum/pulse to the next. All errors of the accelerating system must satisfy the very close tolerances, which make it possible to avoid the exaggerated loss of the intensity of beam upon the acceleration and to obtain the sufficiently narrow spectrum of the distribution of the accelerated particles according to the output energy of linear accelerator. In the adjustable allowances is always important only first significant digit; therefore the calculation of allowances usually can be carried out with essential simplification of the problem.

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Let us examine methodlogy of the calculation of allowances for different perturbation sources. Let us assume that the accelerating system is the sequence of drift tubes loading a volumetric resonator.

a. Random errors in the accelerating system.

Let us examine the random disturbances of small longitudinal vibrations in the nonrelativistic approximation/approach. The amplitudes of the oscillations of phase and relative difference in the impulses/momenta/pulses in this approximation/approach are connected, according to expression (1.87), relationship/ratio

$$g_a = \frac{\Omega}{\omega}\Phi. \tag{1.103}$$

The equation of phase trajectory on plane ψ, g takes the form

$$g_a^2 = g^2 + \left(\frac{\Omega}{\omega}\right)^2 \psi^2. \tag{1.104}$$

In the presence of the random errors in the accelerating system of synchronous particle, generally speaking, there does not exist. Let us examine the longitudinal vibrations of particles relative to synchronous particle in the appropriate ideal system. Let in certain period of the accelerating structure the center of clearance be displaced to value δz relative to regular position. Then particle passes center in the phase of field, which delays relative to correct phase to value

$$\delta\psi = \omega \frac{\delta z}{v_s} = 2\pi \frac{\delta z}{\beta\lambda}.$$

Further, a regular increase in the impulse/momentum/pulse in the accelerating clearance is equal

$$\Delta p = \frac{eE_0TL}{v_s} \cos\varphi.$$

Divergence from the regular increase with the random error in the field and in the phase will comprise

$$\delta p = \Delta p \left(\frac{\delta E_0}{E_0} - \operatorname{tg}\varphi_s \delta\psi\right).$$

Hence, utilizing relationship/ratio (1.26), we obtain

$$\delta g = \frac{kW_\lambda}{\beta} \left(\frac{\delta E_0}{E_0} - \operatorname{tg}\varphi_s \delta\psi\right). \qquad (1.105)$$

Subsequently we will consider the random errors in field and relative attitude of center as the independent variables, distributed in all periods of the accelerating structure according to one and the same probability law with the mathematical expectation, equal to zero. The disturbances/perturbations of impulse/momentum/pulse (1.105) depend on the number of period; the disturbance/perturbation is the greater, the lower the particle speed in the period.

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Differentiating equality (1.104) and supplying/delivering/feeding into into the differential

$$g = g_a \sin \Psi; \quad \psi = \Phi \cos \Psi,$$

it is obtained connection/communication between a change in the amplitude of the oscillations and the disturbances/perturbations of the instantaneous values of values ψ, g in the n-th period:

$$\delta g_{an} = \sin \Psi_n \delta g_n + \left(\frac{\Omega}{\omega}\right)_n \cos \Psi_n \delta \psi.$$

Hence it is apparent that for any particle the mathematical expectation of the disturbance/perturbation of amplitude after this period is equal to zero: $\overline{\delta g_{an}} = 0$. A random increment in the amplitude in the n period subsequently adiabatically attenuates. Toward the end of the accelerator this increment, according to expression (1.94), will give the following contribution to the amplitude

$$\delta g_{an}^{\kappa} = \delta g_{an} \left(\frac{\beta_n}{\beta_\kappa}\right)^{3/4},$$

where β_n— particle speed in the n period; β_κ— particle speed at the output of the accelerating system. After the N periods of the accelerating structure total increment in the amplitude is equal to

$$\Delta g_a = \sum_{n=0}^{N} \delta g_{an} \left(\frac{\beta_n}{\beta_\kappa}\right)^{3/4}.$$

It is obvious that $\overline{\Delta g_a}=0$. Thus, the mathematical expectation of an increment in the amplitude of the longitudinal vibrations of each particle after the flight/span through the accelerator under the assumptions accepted is also equal to zero. However, since the disturbances/perturbations of amplitude in each period are by chance, net gain in the amplitude is distributed according to certain probability law. Let us determine allowances for the random errors in the accelerating system, after restricting possible dispersion Δg_a, i.e. being assigned by value $\sqrt{\overline{(\Delta g_a)^2}}$. The rms value of any random variable x let us designate by the curly brace

$$\langle x\rangle = \sqrt{\overline{x^2}}. \qquad (1.106)$$

Let us assume that the errors of the accelerating systems in different periods structure are independent. Then

$$\langle \Delta g_a\rangle^2 = \sum_{n=0}^{N} \langle \delta g_{an}\rangle^2 \left(\frac{\beta_n}{\beta_K}\right)^{3}.$$

The square of the disturbance/perturbation of amplitude depends on the instantaneous frequency and on the phase of the longitudinal vibrations of the given particle in the period

$$(\delta g_{an})^2 = \sin^2\Psi_n (\delta g_n)^2 + \left(\frac{\Omega}{\omega}\right)_n^2 \cos^2\Psi_n (\delta\psi)^2 + \left(\frac{\Omega}{\omega}\right)_n \sin 2\Psi_n \delta g_n \delta\psi.$$

in question.

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The disturbances/perturbations of amplitude for different particles prove to be different. We will consider that the region within the separatrix is completely filled with particles, so that all phases of longitudinal vibrations are equally probable, and averaged the square of the disturbance/perturbation of amplitude on all particles. Since in this case

$$\overline{\sin^2 \Psi_n} = \overline{\cos^2 \Psi_n} = \frac{1}{2}; \quad \overline{\sin 2\Psi_n} = 0,$$

that we will obtain

$$\langle \Delta g_a \rangle^2 = \frac{1}{2} \sum_{n=0}^{N} \left[\langle \delta g_n \rangle^2 + \left(\frac{\Omega}{\omega} \right)_n^2 \langle \delta \Psi \rangle^2 \right] \left(\frac{\beta_n}{\beta_K} \right)^{5/2}.$$

According to expressions (1.94), (1.105),

$$\left(\frac{\Omega}{\omega} \right)_n = \left(\frac{\Omega}{\omega} \right)_K \left(\frac{\beta_n}{\beta_K} \right)^{-1/2};$$
$$\langle \delta g_n \rangle = \langle \delta g_K \rangle \left(\frac{\beta_n}{\beta_K} \right)^{-1}.$$

Hence

$$\langle \Delta g_a \rangle^2 = \frac{1}{2} \langle \delta g_K \rangle^2 \sum_{n=0}^{N} \left(\frac{\beta_n}{\beta_K} \right)^{1/2} + \frac{1}{2} \left(\frac{\Omega}{\omega} \right)_K^2 \langle \delta \psi \rangle^2 \sum_{n=0}^{N} \left(\frac{\beta_n}{\beta_K} \right)^{3/2}.$$

Since a velocity increment in each period of structure is small, is possible in the latter/last expression to replace sums with the integrals:

$$\langle \Delta g_a \rangle^2 = \frac{1}{2} \langle \delta g_K \rangle^2 \int_0^N \left(\frac{\beta_n}{\beta_K} \right)^{1/2} dn + \frac{1}{2} \left(\frac{\Omega}{\omega} \right)_K^2 \langle \delta \psi \rangle^2 \int_0^N \left(\frac{\beta_n}{\beta_K} \right)^{3/2} dn.$$

From relationship/ratio (1.27) it follows

$$\beta_n = \beta_\kappa - (N-n)\,kW_\lambda.$$

Substituting β_n into the integrals and producing integration, finally we obtain

$$\langle \Delta g_a \rangle = \sqrt{\frac{N}{2}\left[\langle \delta g \rangle^2 - \left(\frac{\Omega}{\omega}\right)^2_N \langle \delta\psi \rangle^2\right]}, \qquad (1.107)$$

where

$$\langle \delta\psi \rangle = 2\pi \left\langle \frac{\delta z}{\beta\lambda} \right\rangle;$$

$$\langle \delta g \rangle = \frac{kW_\lambda}{\beta_N} \sqrt{\left\langle \frac{\delta E_0}{E_0} \right\rangle^2 + 4\pi^2 \operatorname{tg}^2 \varphi_s \left\langle \frac{\delta z}{\beta\lambda} \right\rangle^2}. \qquad (1.108)$$

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Into formulas (1.107), (1.108) are introduced some equivalent values of the frequency of longitudinal vibrations and particle speed

$$\left(\frac{\Omega}{\omega}\right)^2_N = \left(\frac{\Omega}{\omega}\right)^2_\kappa f_1\left(\frac{\beta_0}{\beta_\kappa}\right);$$

$$\frac{1}{\beta_N^2} = \frac{1}{\beta_\kappa^2} \cdot f_2\left(\frac{\beta_0}{\beta_\kappa}\right). \qquad (1.109)$$

Functions f_1 and f_2 exist .

$$f_1(x) = \frac{2}{5} \cdot \frac{1-x^{5/2}}{1-x};$$

$$f_2(x) = \frac{2}{3} \frac{1-x^{3/2}}{1-x}. \qquad (1.110)$$

Values Ω_N, β_N are not the average/mean values of the corresponding parameters at the length of accelerator. From expressions (1.109), (1.110) it is evident that

$$\Omega_{\text{Y}} < \Omega_{\text{K}}; \quad \beta_{\text{Y}} > \beta_{\text{K}}.$$

If the contribution of all disturbances/perturbations to total increment in the amplitude toward the end of the accelerator was identical and equal to the contribution of disturbances/perturbations in the latter/last period, then in formulas (1.107), (1.108) would prove to be values Ω/ω and β, which correspond to the end/lead of the accelerator. As noted above, the disturbances/perturbations of amplitude in the first periods, other conditions being equal, are more than on the latter. However, initial disturbances attenuate so rapidly that their contribution toward the end of the accelerator proves to be smaller than the contribution of the disturbances/perturbations, which occur in the latter/last periods of the accelerating structure.

With the change of argument $x=\frac{\beta_0}{\beta_{\text{K}}}$ from zero to unity, function $f_1(x)$ and $f_2(x)$ are changed in the limits

$$\frac{2}{5} \leqslant f_1(x) \leqslant 1; \quad \frac{2}{3} \leqslant f_2(x) \leqslant 1.$$

For the creation of certain supply in the allowances it is possible to accept $f_1=f_2=1$. Then into formulas (1.107), (1.108) enter values Ω and β at the end of the accelerating system.

From formulas (1.107), (1.108) it is evident that the disturbances/perturbations of amplitude with the assigned errors in the field and in the arrangement of the accelerating clearances grow/rise with an increase in the multiplicity of the period of structure. Therefore the high multiplicity is undesired and can be caused only by serious design considerations. Most frequently they select k=1.

As it was established/installed, the vertical spread/scope of separatrix attenuates slower than the amplitude of oscillations g_A. The permissible value of the rms value of an increment in the amplitude can be selected from the condition so that the particles in the process of acceleration would not exceed the limits of the region, limited by separatrix.

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However, requirements for the monochromaticity of the accelerated beam can make it necessary to dwell on lower values Δg_A. For evaluating the order of the appearing allowances let us examine proton linear accelerator with the parameters, stipulated above: k=1; $W_\lambda = 2.7 \cdot 10^{-3}$; $\cos\varphi_s = 0.8$; $W_0 = 700$ keV; $W_K = 24$ MeV. We have $\beta_0 = 0.04$; $\beta_K = 0.22$; $\left(\frac{\Omega}{\omega}\right)^2_K = 1.5 \cdot 10^{-3}$. The substitution of values into formulas (1.107), (1.108) gives

$$\left\langle\frac{\delta E_0}{E_0}\right\rangle^2-400\left\langle\frac{\delta z}{\beta\lambda}\right\rangle^2\approx 175\,(\Delta g_a)^2.$$

Let us accept the initial amplitude of the longitudinal vibrations of the separatrix equal to vertical spread/scope at the input: $g_a^0\approx g^0_{\text{макс}}=0.073$. At the output of accelerator $g_a^\kappa\approx 0.008$. Let us establish/install allowances in such a way that the random errors in the accelerating system would increase the scatter of particles in on the impulses/momenta/pulses at the output not more than by 25o/o. Then $\Delta g_a=0.002$. Distributing allowances evenly between both basic errors, we obtain

$$\left\langle\frac{\delta E_0}{E_0}\right\rangle\approx 2\%;$$
$$\left\langle\frac{\delta z}{\beta\lambda}\right\rangle\approx 0.1\%.$$

B. Systematic errors in the accelerating system.

Systematic we will call the errors whose distribution along the axis of accelerator does not carry random character. Such errors, besides a difference in the established/installed amplitude of accelerating field from a precise value/significance, include the calculated errors in determination of the factor of transit time and lengths of the periods of structure, and also scale errors during the longitudinal arrangement of drift tubes.

Let us examine the accelerating system, assembled in a precise conformity with calcultion, without the random errors. By this is already assigned an increase in the length of each following period of the structure of relatively of preceding/previous, i.e., is assigned a partial increase in energy of synchronous particle in each this period. At fixed value/significance λ

$$\Delta L = k\lambda\Delta\beta,$$

where

$$\Delta\beta = \frac{\Delta W_s}{\beta\gamma^3\mathscr{E}_0}.$$

Hence, according to expression (1.10),

$$\Delta L = E_0 T\lambda^2 \cos\varphi_s \frac{ek^2}{\gamma^3\mathscr{E}_0} \quad \text{const.} \qquad (1.111)$$

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In the nonrelativistic approximation/approach γ~~1 and, therefore,

$$E_0 T\lambda^2 \cos\varphi_s = C_0 = \text{const.} \qquad (1.112)$$

A change in any of three values E_0, T, λ causes the mixing of the synchronous phase

$$\Delta\varphi_s = \operatorname{ctg}\varphi_s \left(\frac{\Delta E_0}{E_0} + \frac{\Delta T}{T} + 2\frac{\Delta\lambda}{\lambda}\right). \qquad (1.113)$$

In this case specific acceleration in the already assembled accelerator is changed only with a change in the wavelength of

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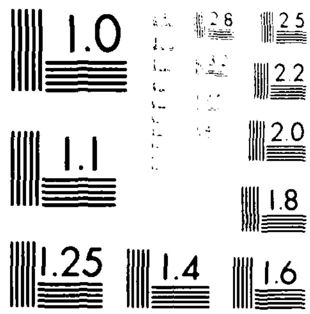
1.0
2.8
2.5
2.2
2.0
1.1
1.8
1.25
1.4
1.6

accelerating field. Utilizing equalities (1.25), (1.112), we will obtain

$$W_\lambda = \frac{eC_0}{\varepsilon_0} \cdot \frac{1}{\lambda}. \qquad (1.114)$$

In accordance with equality (1.114) and final energy of synchronous particle depends only on the wavelength of accelerating field. If specific acceleration is constant along the length of accelerator, then

$$W_s = \frac{eC_0}{\lambda^2} l_v. \qquad (1.115)$$

where l_v — the path length of acceleration. A change in the parameter λ can be connected with the real frequency switch of accelerating field and with the error of scale during the arrangement of drift tubes along the axis. For example, if due to the error with the comparison of tape measure each scale division exceeds rating, then the true value/significance λ, which is determining the real arrangement of drift tubes, will be less than the calculated, and final energy of synchronous particle will prove to be above calculated. Frequency stability of accelerating field always can be made sufficiently high due to the quartz-crystal control.

The given considerations prove to be especially demonstrative, if we examine particle acceleration in the equivalent traveling wave. It is obvious that final energy of synchronous particle depends only on the speed of the traveling wave, which seizes particles, but it does not depend on the amplitude of the traveling wave. In the relativistic approximation/approach the picture qualitatively is not changed, but the dependence of synchronous phase on the amplitude of field and factor of transit time becomes more complicated; according to expression (1.111)

$$E_0T\lambda^2 \cos \varphi_s (\mathscr{E}_0 - W_0 - eE_0Tl_y \cos \varphi_s)^{-3} = \mathrm{const}$$

Let us note that although energy of synchronous particle changes only with the variation of the parameter λ, the distribution of all particles according to the energy spectrum depends on amplitude and "inclination/slope" of field. Therefore, changing field, it is possible in certain cases to move the maximum of energy spectrum in the limits of separatrix.

The displacement of synchronous phase, connected with the departure/attendance of the amplitude of field from the nominal value, produces change in the frequency of longitudinal vibrations (1.62) and phase factor (1.98).

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The requirement of the stability of phase change of longitudinal vibrations at the length of the accelerator superimposes the close tolerance for the retention/preservation/maintaining of the rating value of field. We will use formulas (1.99). Since φ=const,

$$\frac{\delta(\Delta\Psi)}{\Delta\Psi} = \frac{\delta x}{x}.$$

For evaluating the allowance in the field it suffices to accept nonrelativistic approximation/approach (1.113), since basic phase change occurs with low energies. As a result we obtain

$$\frac{\delta(\Delta\Psi)}{\Delta\Psi} = \frac{1}{2\sin^2\varphi_s} \cdot \frac{\delta E_0}{E_0}. \qquad (1.116)$$

With an increase in the absolute value of synchronous phase the dependence of phase change of longitudinal vibrations on the amplitude of field is weakened/attenuated. On the contrary, at the low absolute values of synchronous phase raid susceptibly/critically depends on the amplitude of field. Let us examine a numerical example. When $W_\lambda = 2.7 \cdot 10^{-3}$ and $\cos\varphi_s = 0.7864$ from the wave energy 700 keV to final energy 100 MeV in the linear by proton accelerator occur 6.5 longitudinal vibrations. Being assigned $\delta(\Delta\Psi) \approx 15^\circ$, we obtain $\frac{\delta E_0}{E_0} = 0.5\%$.

Let us rate/estimate the effect of the rejected/thrown corrections of the second approximation/approach to factor of transit time and length of the period of the accelerating structure (see S1.1). Correction to the factor of transit time is determined by formula (1.20):

$$\frac{\Delta T}{T} = -\frac{\alpha k}{6}\chi, \qquad (1.117)$$

moreover in accordance with expressions (1.21), (1.62)

$$\chi \approx \pi^2 \left(\frac{\Omega}{\omega}\right)^2.$$

Thus, correction (1.117) has maximum value with the energy of injection, and then virtually monotonically decreases. Let us examine the knowingly worse case, after assuming that the factor of transit time is changed in the beginning of accelerator by jump. This will give synchronous phase jump to the value, determined by equality (1.13),

$$\Delta\varphi_s = \operatorname{ctg}\varphi_s \left(\frac{\Delta T}{T}\right)_0.$$

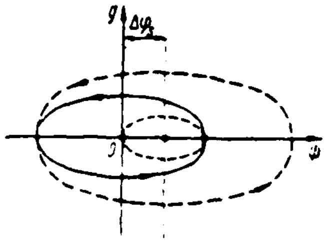

Fig. 1.13.

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The "instantaneous" synchronous phase jump increases the amplitude of small phase oscillations by the value of jump (Fig. 1.13) and respectively increases the amplitude of the oscillations of the impulses/momenta/pulses:

$$\Delta g_a = \frac{\Omega}{\omega}\Delta\varphi_s \, . \qquad (1.118)$$

Equality (1.118) follows from relationship/ratio (1.103). As a result we obtain

$$\Delta g_a = \frac{\pi^2 ak}{6} \operatorname{ctg}\varphi_s \left(\frac{\Omega}{\omega}\right)^3 .$$

The numerical this magnitude estimate shows that the effect of the correction of the second approximation/approach is negligibly small. Analogous result is obtained during the estimation of correction to the period of the accelerating structure.

c. Damage of the regular structure of the accelerating system.

The essential disturbances/breakdowns of the regularity of high-frequency system are possible in by multi-resonator accelerator on the transitions between resonators. If high-frequency fields in the adjacent resonators are shifted relative to each other on the phase value $\Delta\varphi$, then this is equivalent to the shift/shear of synchronous phase $\Delta\varphi_s = \Delta\varphi$ and is called an increase in the amplitude of the oscillations of impulses/momenta/pulses (1.118). A similar effect appears, if fields in the adjacent resonators are cophasal, but they are distinguished by amplitude; according to expressions (1.113), (1.118), we have

$$\Delta g_a = |\operatorname{ctg}\varphi_s| \frac{\Omega}{\omega} \cdot \frac{\Delta E_0}{E_0}. \quad (1.119)$$

The allowance for the stability of field in each resonator is the stricter, the lower the energy of particles and the less the absolute value of synchronous phase. This allowance usually proves to be wider than the allowance, determined by the stability of phase change of longitudinal oscillations (1.116).

In many instances the resonators of linear accelerator it is necessary to divide according to the design considerations the idle gaps/intervals, free from accelerating field. Idle gaps/intervals are introduced for positioning/arranging of vacuum locks, stations of the beam monitoring, etc. Regardless of the fact, are terminated resonators with half-tubes or half-gaps, idle gap/interval always leads to the supplementary drift of particles, which distorts the phase volume of small longitudinal vibrations. Fig. 1.14 gives in coordinates $\psi. g$ the phase volume of the beam before the idle gap/interval. Volume is limited to ellipse 1 whose semi-axes coincide with the coordinate axes. The section of the drift between the resonators distorts the initial ellipse (curve 2), since each particle, retaining impulse/momentum/pulse, is displaced on the phase relative to field to the value proportional $-\dot{\psi} \div g$.

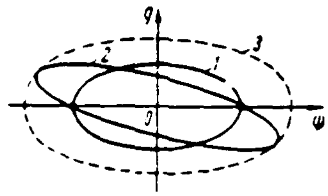

Fig. 1.14.

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Dotted line (curve 3) designated the greatest phase trajectory in the following after idle gap/interval resonator. The region, occupied with the representative points of beam, rotates with the frequency of longitudinal vibrations within the ellipse (curve 3). Therefore, clusters after idle gap/interval begin to fluctuate.

Another source of the emergence of the pulsations, which lead to an increase in the amplitude of oscillations of phases or impulses/momenta/pulses, is a possible difference in the specific accelerations in the adjacent resonators. The difference in the values of specific accelerations causes the jump of the frequency of longitudinal vibrations on the transition. The case when frequency at the output of the first resonator Ω_1 higher than frequency at the input of the following Ω_2, is shown in Fig. 1.15a, and 1.15b - the case $\Omega_2 > \Omega_1$. Solid line limited the phase volume of cluster to the transition. Dotted curves - phase trajectories of different particles after transition.

Let l - length of idle gap/interval; ψ_1, g_1 — the coordinate of certain particle at the input of idle gap/interval. The representative point of this particle lies/rests on the ellipse, which limits phase volume,

$$\Phi_1^2 = \psi_1^2 - \left(\frac{\omega}{\Omega_1}\right)^2 g_1^2.$$

Coordinates of particle at the output of the idle gap/interval

$$\psi_2 = \psi_1 + \dot{\psi}\frac{l}{v_\phi}; \quad g_2 = g_1. \qquad (1.120)$$

In nonrelativistic approximation/approach $\frac{d\psi}{dt} = -\omega g$ (see 1.45a). Point ψ_2, g_2 lies/rests on certain ellipse

$$\Phi_2^2 = \psi_2^2 + \left(\frac{\omega}{\Omega_2}\right)^2 g_2^2. \qquad (1.121)$$

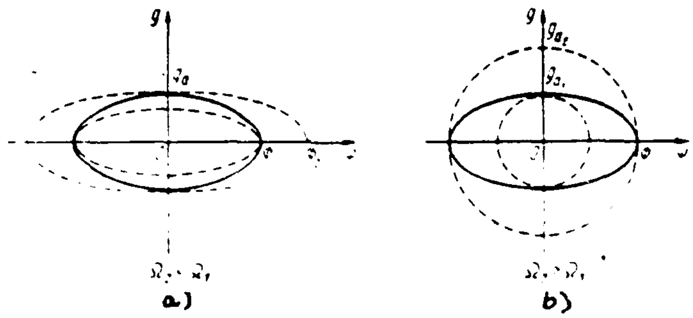

Fig. 1.15.

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Let us supply in equation (1.121) values ψ_2, g_2 from equations (1.120) and will find the pair of coordinates ψ_1, g_1, for which amplitude Φ_2 it has the greatest value/significance. For this let us represent ψ_1, g_1 in the form

$$\psi_1 = \Phi_1 \sin \Psi; \quad g_1 = \frac{\Omega_1}{\omega} \Phi_1 \cos \Psi$$

and let us find value/significance Ψ, corresponding to extremum Φ_2. As a result we obtain

$$\frac{\Phi_2}{\Phi_1} = \frac{1}{\sqrt{2}} \left[1 + \left(\frac{\Omega_1}{\Omega_2} \right)^2 + a^2 \pm \right.$$
$$\left. \pm \sqrt{\left[1 + \left(\frac{\Omega_1}{\Omega_2} \right)^2 + a^2 \right]^2 - 4 \left(\frac{\Omega_1}{\Omega_2} \right)^2} \right]^{1/2}; \qquad (1.122)$$
$$g_{a_2} = g_{a_1} \frac{\Omega_2}{\Omega_1} \cdot \frac{\Phi_2}{\Phi_1},$$

where

$$a = 2\pi \frac{l}{\beta\lambda} \cdot \frac{\Omega_1}{\omega}. \qquad (1.123)$$

If $\Omega_1=\Omega_2$, then relationships/ratios (1.122) are reduced to the simpler

$$\frac{\Phi_2}{\Phi_1}=\sqrt{1-\frac{a^2}{2}\pm a\sqrt{1-\frac{a^2}{4}}};\quad \frac{g_{a_2}}{g_{a_1}}=\frac{\Phi_2}{\Phi_1}. \quad (1.124)$$

Usually $\alpha<<1$; in this case

$$\frac{\Phi_2}{\Phi_1}\approx 1\pm\frac{a}{2}; \quad (1.124a)$$

The effect of idle gap/interval on the energy scatter of particles is the less, the higher the energy of particles upon transfer. However, even with comparatively high energies idle gaps/intervals noticeably affect the parameters of beam. Thus, output energy of the first resonator of the linear accelerator of protons I-100 is 43 MeV; the frequency of longitudinal vibrations in this place Ω~~0.03ω. With $l=2\beta\lambda$ the amplitude of longitudinal oscillations increases by 20o/o.

In the absence of the idle clearance a=0 and formulas (1.122) lead to two cases

$$1)\ \Omega_2<\Omega_1;\ g_{a_2}=g_{a_1};\quad \Phi_2=\frac{\Omega_1}{\Omega_2}\Phi_1;$$

$$2)\ \Omega_2>\Omega_1;\ g_{a_2}=\frac{\Omega_2}{\Omega_1}g_{a_1};\quad \Phi_2=\Phi_1.$$

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Chapter 2.

Transverse vibrations of particles in beams with the negligible density of space charge.

§2.1. Defocusing factors in the linear accelerator.

In the linear accelerators must be provided special measures for the beam focusing of the charged/loaded particles. It is necessary to create either the supplementary permanent fields, intended for the retention of particles near the axis of accelerator, or the special configurations of the high-frequency field, which permit implemention of simultaneously acceleration and focusing. If such special measures are absent, then beam of particles diffuses. It is possible to name/call three basic factors, that lead to the defocusing of beam in the linear accelerator: 1) the disordered scatter of transverse thermal particle speed; 2) the defocusing action of accelerating field; 3) electrostatic pushing apart between the similarly/analogously charged/loaded particles of beam. Let us discuss the first two factors. Latter/last factor is in detail examined in Chapter 3.

The particles of beam possess disordered component of velocity on all three coordinates, caused by thermal particle motion. Particle distribution according to the thermal velocities depends on temperature and configuration of source. If the disordered component longitudinal velocity it is possible to disregard in view of the fact that it usually to is many orders less than the regular component, then disordered components of transversing speeds substantially affect the behavior of beam in the accelerator. Specifically, due to the presence of the transverse thermal velocities beam of particles in the accelerator or in the ion guide never succeeds in gathering in the point focus. Since there are always thermal velocities, directed outside of beam, the latter diffuses during the free drift of particles [30]. Due to the scatter of particles on the speeds and on the attitude the representative points of beam occupy final volume in the six-dimensional phase space. In accordance with Liouville's theorem six-dimensional phase volume is invariant in the space of canonical-conjugated/combined variable/alternating.

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Will examine the projections of six-dimensional phase volume on the phase planes z, p_z (or ψ, p_ψ); x, p_x; y, p_y. In the general case

Liouville's theorem does not require the invariance of each projection. However, a change in two graduated phase volume on one of the planes produces change its on other planes due to the invariance of six-dimensional volume. There is an important special case when variable/alternating in the equations of motion are divided. Then Liouville's theorem proves to be valid for each subspace individually. Let us examine the character of transverse particle motion in any plane (for example, XOZ) depending on the form of the two-dimensional phase volume of beam in coordinates x, x'=dx/dz.

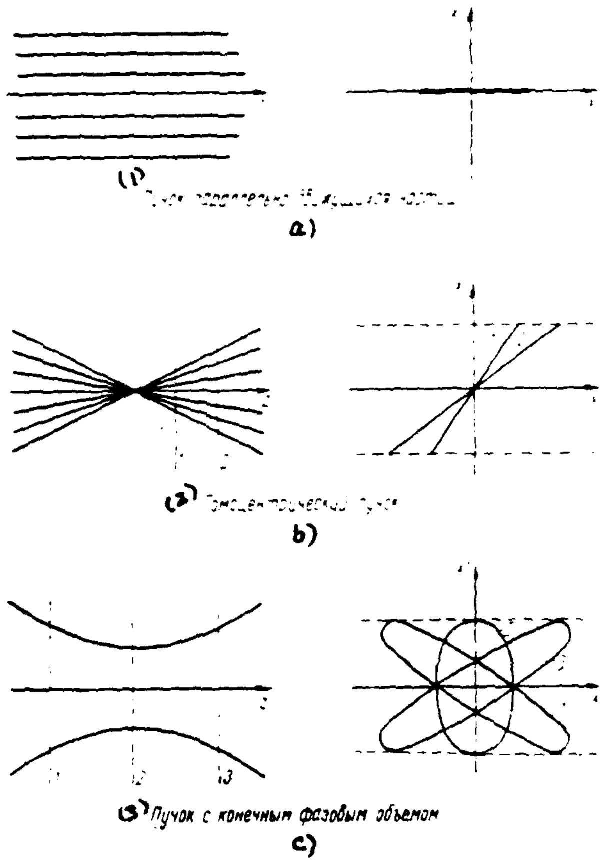

Fig. 2.1.

Key: (1). Beam of the in parallel moving/driving particles. (2). Homocentric ray. (3). Beam with finite phase volume.

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In Fig. 2.1a all particles of beam move in parallel to axis and, therefore, they have zero transverse component of speed. In accordance with this the representative points lie/rest on the axis of abscissas and occupy the cut, equal along the length to the diameter of beam. The phase volume of beam is equal to zero. In Fig. 2.1b particles move over straight paths, which converge to one point (homocentric ray). The transversing speeds of particles in this beam are proportional to misalignment. The representative points with the assigned longitudinal coordinate z prove to be in the rectilinear cut, inclined toward the axis of abscissas. To particles in section 1 (see Fig. 2.1b to the left) correspond the representative points in cut 1 (see Fig. 2.1b to the right). The projection of intercept on an axis of abscissas is equal to the diameter of beam. With further particle motion of the cuttings off it rotates clockwise and it is lengthened (see cut 2). The phase volume of beam remains equal to zero. To convergent beam corresponds cut with the negative inclination/slope. Homocentric ray with the aid of the adequate/approaching lenses always can be converted into the beam parallel to the moving particles. On the other hand, lens with the nonlinear fields is converted rectilinear cut on the phase plane into the curve, retaining zero phase volume. Fig. 2.1c gives envelope of particles with the phase volume of finite quantity. In this beam at each given distance from the axis can be located the particles whose speeds are continuous in certain finite interval of velocities. Let the phase volume on plane x, x' be limited by ellipse. To convergent beam corresponds ellipse with the negative inclination/slope of major axis (curve 1); to divergent beam - ellipse with the positive inclination/slope of major axis (curve 3). The projection of ellipse on the axis of abscissas is equal to the diameter of beam at the particular point z. During the free drift of particles the phase volume of convergent beam rotates clockwise, since all representative points move in the positive (with x'>0) or negative (with x'<0) direction along the axis of abscissas, without changing transversing speed. At certain point z the major axis of the ellipse coincides with the axis of ordinates (curve 2), which corresponds to the crossover of bundle. At the point of crossover envelope of particles is parallel to longitudinal axis. Subsequently, after crossover, the beam with the final phase volume diffuses even in the absence of any other defocusing factors.

With the particle motion in electromagnetic field the generalized momentum, canonically conjugated/combined with the Cartesian coordinates, is the variable/alternating [21]

$$\mathbf{P}=\mathbf{p}+e\mathbf{A}, \qquad (2.1)$$

where p=mv - particle momentum; A - vector potential of field; B=rot A. If in the magnetic field is absent longitudinal component $(B_z=0)$, then $A_x=A_y=0$ and $P_x=p_x$; $P_y=p_y$. Thus, during the motion in plane XOZ, which does not depend on the remaining variable/alternating, and when

$B_z = 0$ phase volume is retained in variable/alternating x, p_x.

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Let us name/call the two-dimensional transverse phase volume of beam value

$$V_n = \frac{1}{\pi m_0 c} \int dx\, dp_x. \qquad (2.2)$$

where the integral is taken by entire volume, occupied by the representative points of beam on plane x, p_x. In other words, V_n -- divided on π the area, occupied by beam, on the plane displacement - given impulse/momentum/pulse.

Frequently as the measure of the phase volume of beam is considered divided on π the area, occupied with the representative points of beam on plane x, x'=dx/dz:

$$E = \frac{1}{\pi} \int dx\, dx'. \qquad (2.3)$$

This value is called the emittance of beam. The emittance of beam is connected with the transverse phase volume with the relationship/ratio

$$E = \frac{V_n}{\beta\gamma}. \qquad (2.4)$$

Emittance is convenient as the value, usually directly determined from the direct measurements. Furthermore, emittance is conveniently used, as this will be clearly from the following, in the theory of the circular accelerators. However, in the theory of linear accelerators to more preferably use the concept of transverse phase volume (2.2) as by the value, not energy-dependent of particles. Emittance is determined on the plane of the variable/alternating, which are not canonical-conjugated/combined. With an increase in the energy of particles the emittance of beam decreases and it vanishes with $\gamma \rightarrow \infty$. Respectively it vanishes scatter of path inclinations. With the low energies the decrease of the scatter of trajectories is connected in essence only with an increase in the longitudinal velocity of particles. With the relativistic energies the dominant role begins to play an increase in Lorenz's factor. The decrease of the scatter of trajectories with an increase in the energy of particles leads to the fact that the beam of relativistic particles in the short sections of drift does not need focusing. One should, however, note that with the distance of drift in several ten meters has already been perceived the deliquescence even high-energy electron beam.

The particles, seized into acceleration mode by the axisymmetric high-frequency field, test/experience in this field transverse defocusing. Defocusing occurs under specific conditions, which are usually satisfied in practice. Actually/really, the equation of motion of particles in the field takes in the general case the form

$$\frac{d\mathbf{p}}{dt}=e\mathbf{E}+e[\mathbf{vB}]. \qquad (2.5)$$

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Let us accept cylindrical coordinate system r, φ, z with z axis, directed along the axis of accelerator, and will examine the axisymmetric field, type TM, $E_\varphi=0;\ B_r=B_z=0.$ Then

$$\frac{dp_r}{dt}=e(E_r-vB_\varphi),$$

where v - the longitudinal velocity of particle. As it is easy to obtain,

$$\frac{dp_r}{dt}=m_0\gamma\left(\frac{d^2r}{dt}+\frac{dr}{dt}\cdot\frac{d}{dt}\ln\gamma\right). \qquad (2.6)$$

moreover Lorenz's factor is determined only by the longitudinal velocity of particle, since the transverse components of the velocity substantially lower than longitudinal. Coefficient with first-order derivative in the right side of equation (2.6) is low, since ln γ - slow function of time. If we examine by the component/term/addend in the right side of equation (2.6) it is possible to disregard. Detailed estimations show that at the usually utilized in the ionic accelerators values of specific acceleration the effect of the rejected term on the particle motion is negligibly small up to γ~~2. Thus,

$$\frac{d^2r}{dt^2}=\frac{e}{m_0\gamma}\left(E_r-vB_\varphi\right). \qquad (2.7)$$

Further, for the points, close to the axis, from the equations of Maxwell

$$\frac{E_r}{r}+\frac{\partial E_r}{\partial r}+\frac{\partial E_z}{\partial z}=0;\quad \frac{B_\varphi}{r}+\frac{\partial B_\varphi}{\partial r}=\frac{1}{c^2}\cdot\frac{\partial E_z}{\partial t} \qquad (2.8)$$

considering that on axis $E_r=B_\varphi=0$, follows

$$E_r=-\frac{1}{2}\cdot\frac{\partial E_z}{\partial z}r;\quad B_\varphi=\frac{1}{2c^2}\cdot\frac{\partial E_z}{\partial t}r. \qquad (2.9)$$

Hence

$$\frac{d^2r}{dt^2}=-\frac{e}{2m_0\gamma}\left(\frac{\partial E_z}{\partial z}+\frac{v}{c^2}\cdot\frac{\partial E_z}{\partial t}\right)r. \qquad (2.10)$$

Let us examine, first of all, particle motion in the field of traveling wave (1.43). According to equation (1.43),

$$\frac{\partial E_z}{\partial t}=-v_s\frac{\partial E_z}{\partial z}; \qquad (2.11)$$

$$\frac{\partial E_z}{\partial z}=\frac{\omega}{v_s}E\sin\varphi. \qquad (2.12)$$

Substituting equalities (2.11), (2.12) in equation (2.10) and taking into account designation (1.25a), (1.62), (1.82), for the paraxial particles, close to synchronous $(\varphi\approx\varphi_s;\ v\approx v_s)$, we obtain

$$\frac{d^2r}{dt^2}-\frac{1}{2}\Omega^2r=0. \qquad (2.13)$$

Longitudinal vibrations in the linear conservative approximation/approach are described by the equation, which follows from expression (1.72)

$$\frac{d^2\psi}{dt^2} + \Omega^2\psi = 0. \qquad (2.14)$$

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From equations (2.13), (2.14) it is evident that the stability conditions for longitudinal and transverse vibrations in the traveling wave are incompatible. If sin $\varphi_s < 0$, then $\Omega^2 > 0$ and the longitudinal vibrations of particles are stable, but transverse vibrations prove to be unstable.

The defocusing action of accelerating field, just as the defocusing action of the scatter of thermal velocities, decreases with an increase in the energy of the particles: with $\beta \to 1$ we have $\Omega^2 \to 0$. This is explained by the fact that the magnetic component of high-frequency field in contrast to electrical focuses particles, moreover with an increase in phase wave velocity the action of magnetic field is intensified. In the limit, with $\beta \to 1$,. effect of both components of high-frequency fields to the transverse particle motion is compensated.

Let the particles be accelerated in the field of standing waves, for example in the accelerating system with drift tubes. The sag/sagging accelerating field in the clearance between drift tubes, connected with the edge effects, leads, according to equation (2.9), to the appearance of a radial component of field (Fig. 2.2). Let us recall that the electrostatic field between two diaphragms always acts as the converging lens independent of the direction of field [31]. Actually/really, let us suppose that the field is directed along the particle motion. Then particle obtains at the input of clearance the supplementary transverse impulse, directed toward the axis, and at the output - from the axis. Since the particle speed in the clearance grows/rises, refraction of trajectory at the input proves to be more than at the output, and as a result particle trajectory is inclined axis. When field has opposite direction, then the particle speed in the clearance falls, but the defocusing action occurs at the input of clearance, and that focusing - at the output. In the linear accelerator the field in the clearance is changed in the time, moreover for the particles of those moving near the synchronous, $E_{rx} < E_{rвых}$, so that the defocusing radial component at the output of clearance exceeds the focusing radial component at the entrance. If field is changed sufficiently slowly (which occurs at the low absolute values of synchronous phase), then will prevail the effect of an increase in the particle speed in the clearance and clearance as a whole will be the converging lens as in the electrostatic case. But if field change for the time of flight of the particle along the clearance is sufficiently great, then the effect of a velocity increment does not compensate the difference in action of radial components at the entrance and output and clearance will act as diverging lens.

Equation (2.10) describes particle motion in the accelerating clearance, if E_z — the field, concentrated in the clearance. Let us estabiish connection/communication between derivatives $\frac{\partial E_z}{\partial t}$ and $\frac{\partial E_z}{\partial z}$, being limited to the first running harmonic of Fourier-expansion of field (1.42). Taking into account equality (2.11) we obtain

$$\frac{d^2r}{dt^2} = -\frac{e}{2m_0\gamma^3}\cdot\frac{\partial E_z}{\partial z}r. \qquad (2.15)$$

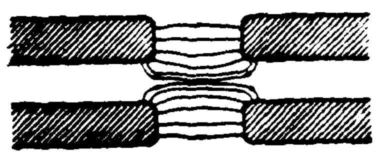

Fig. 2.2.

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Let us rate/estimate, on the basis of equation (2.15), refraction of the trajectory of synchronous particle in the accelerating clearance. In this case we will consider that the particle displacement at the gap length is not changed, and we approximate field in the clearance by "square wave" ($E_g=0$ when $z-z_0 > g\,2$ and $E_g=\text{const}$ when $z-z_0 < g\,2$. Examining equation (2.15) in one period of the accelerating structure, let us pass from differentiation with respect to time to differentiation on the longitudinal coordinate; in the nonrelativistic approximation/approach

$$\frac{d^2r}{dz^2} = -\frac{e}{2m_0v^2}\cdot\frac{\partial E_z}{dz}\,r. \qquad (2.16)$$

Fracture of trajectory at the entrance of the clearance

$$\Delta\frac{dr}{dz} \approx -\frac{e}{2m_0v^2_{\text{вх}}}\,r\int\limits_{-\infty}^{-g/2}\frac{dE_z}{\partial z}\,dz,$$

or

$$\Delta\frac{dr}{dz} \approx -\frac{e}{2m_0v^2_{\text{вх}}}\,rE_{z(\text{вх})}.$$

Analogously at the output of clearance we have

$$\Delta\frac{dr}{dz} \approx \frac{e}{2m_0v^2_{\text{вых}}}\,rE_{z(\text{вых})}.$$

Total refraction of trajectory on the clearance will comprise

$$\Delta\frac{dr}{dz} \approx \frac{er}{2m_0}\left[\frac{E_{z(\text{вых})}}{v^2_{\text{вых}}} - \frac{E_{z(\text{вх})}}{v^2_{\text{вх}}}\right].$$

$$\Delta E_z = E_{z(\text{вых})} - E_{z(\text{вх})};$$
$$\Delta v = v_{\text{вых}} - v_{\text{вх}}.$$

Then in the first approximation,

$$\Delta \frac{dr}{dz} \approx \frac{eE_{z(\text{вх})}}{2m_0 v_{\text{вх}}^2} \left[\frac{\Delta E_z}{E_{z(\text{вх})}} - 2\frac{\Delta v}{v_{\text{вх}}} \right] r. \qquad (2.17)$$

From expression (2.17) it is evident that with satisfaction of the condition

$$\frac{\Delta v}{v} < \frac{\Delta E_z}{2E_z} \qquad (2.18)$$

the accelerating clearance acts on the particles, close to the synchronous as diverging lens. In accordance with the approximation

$$E_{z(\text{вх})} = E_g \cos\left(-\frac{\omega g}{2v} + \varphi_s\right); \; E_{z(\text{вых})} = E_g \cos\left(\frac{\omega g}{2v} + \varphi_s\right). \qquad (2.19)$$

accepted.

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Hence

$$\frac{\Delta E_z}{2E_z} \approx \sin \pi\alpha \; \text{tg}\, \varphi_s,$$

where α - coefficient of clearance (1.23). Taking into account equality (1.27), it is possible to reduce condition (2.18) to the form

$$|\text{tg}\, \varphi_s| > \frac{kW_\lambda}{\beta \sin \pi\alpha}. \qquad (2.18a)$$

If $W_\lambda = 3 \cdot 10^{-3}$; $\beta = 0{,}04$; $\alpha = 1/4$, then the accelerating clearance will work as the converging lens only when $|\text{tg}\, \varphi_s| < 0.1$ or $|\varphi_s| < 6°$. The capture region of particles on the phases proves to be in this case inadmissibly to small. At the virtually acceptable values of synchronous phase the defocusing action of variable field is so great that usually it is possible to disregard the effect of acceleration in the clearance. With an increase in the energy of particles inequality (2.18a) is amplified, which virtually reduces to zero regions of the stable phases in which the accelerating clearance focuses particles.

In the ionic linear accelerators is most frequently utilized the strong focusing by quadrupole lenses. In the electron accelerators and sometimes in proton the particles are focused by longitudinal magnetic field. Structural/design complexity and high cost/value of the technological equipment, intended for the creation of the external focusing fields, makes it necessary to search for the methods of focusing due to the adequate/approaching geometry of accelerating field itself. Historically the first method of beam focusing in the proton linear accelerators was grid focusing [7, 32, 33] which happens to be one of the types of focusing by means of the accelerating field. Grids or foils, which closed the entrance of each drift tube, created the geometry of field necessary for the focusing in the clearances. However, from grid focusing at present they refused due to the exaggerated losses of beam current in the accelerator and due to others, less essential ones, deficiencies/lacks. From other proposed in the different time methods of focusing due to accelerating field is most promising, apparently, the use of high-frequency quadrupole lenses [34]. This method, based on the failure of the axial symmetry of accelerating field, is at present studied theoretically and experimentally [35-37], but practical use/application thus far it did not find.

§ 2.2. Strong focusing. Quadrupole lenses.

In the managements/manuals on electron optics usually in detail is examined the particle motion in the longitudinal magnetic field (optician Busch). The fields of solenoids were utilized for the focusing of electron beams in different areas of technology even long before the appearance of accelerators.

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With the particle focusing of high-energy by longitudinal field basic motion is directed in parallel to magnetic lines of force, so that the appearing forces are proportional to small transverse components of speed. It is obvious that more effective would be the magnetic field, directed perpendicularly to motion. Then the focusing forces would prove to be proportional to the longitudinal velocity of particles. Analogous situation occurs, also, with the particle focusing by the electrostatic fields. It is desirable so that the lines of force of the focusing electrostatic field would be in essence directed perpendicular to longitudinal particle motion.

However, in the space, free from the charges or the currents, it is not possible to form the fields which would create the radial forces, directed toward the axis simultaneously at all angles. In particular, fields with the quadrupole symmetry (Fig. 2.3), directed created toward the axis of force in plane x, simultaneously create the defocusing forces in plane y. Fig. 2.3a depicts magnetic quadrupole, moreover it is assumed that the particles move from the plane of drawing to the reader, while in Fig. 2.3b - electrostatic quadrupole. If we place along the axis quadrupole lenses then so that the adjacent lenses would be turned relative to each other on 90°, then in each of the planes will be alternately created the focusing and defocusing sections. With satisfaction of the specified conditions this system of lenses proves to be focusing. Actually/really, to the particle, which moves accurately along the axis, the forces do not act. The further the particle from the axis, the greater the acting forces. This makes it possible to understand, why two consecutively/serially confronting lenses, expanded/scanned on 90°, can focus in both planes. Let the particle fall first in the focusing section. In this section the particle trajectory is bent towards axis (Fig. 2.4) and particles will pass the defocusing section with the smaller divergence from the axis, than in the first section. The focusing forces prove to be more than defocusing, so that as a whole the pair of quadrupole lenses proves to be accumulating.

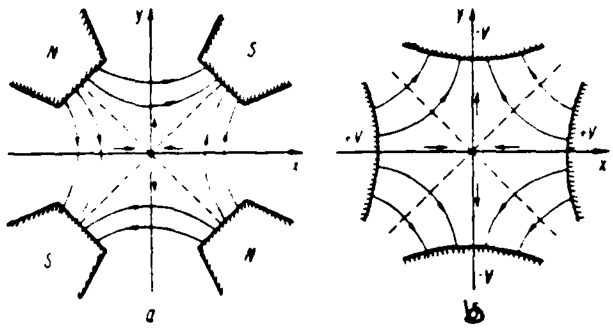

Fig. 2.3.

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A similar effect appears also in such a case, when is at first arranged/located the defocusing section: particle displacement from the axis in the defocusing section is less than on that focusing. In this, speaking in general terms, and consists the idea of hard/rigid, or alternating, focusing. From the given qualitative picture it is evident that with the strong focusing the fundamental value/significance has a presence of field gradient. If field gradients in the quadrupole lenses are too great, then already after the focusing section particle can cross axis and then the defocusing section deflects trajectory in other direction from the axis. The pair of lenses will prove to be scattering.

The strong-focusing channel consists of a large number of consecutively/serially alternating focusing and defocusing sections. In this channel there is a clearly expressed periodicity of structure. The smallest length of the repetition of structure we will call the period of focusing field S. On Fig. 2.5 is schematically shown the particle trajectory in the strong-focusing channel. Trajectory is modulated with the period of focusing field. In all focusing sections the particle on the average is distant from the axis more than on adjacent those defocusing. The stability conditions of oscillations/vibrations in the long channels differ from the conditions, under which the combination of quadrupole lenses F and D, which constitute one period, is accumulating. The object/subject of our further examination in essence will be the theory of the channel optic/optics which is more complicated than the theory of one converging lens.

The period of focusing field can consist of different combinations F- and D-quadrupoles.

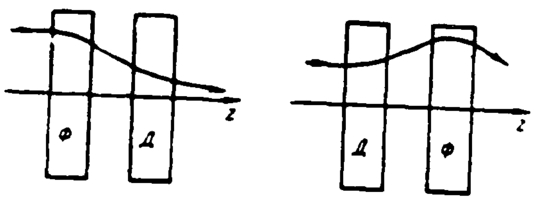

Fig. 2.4.

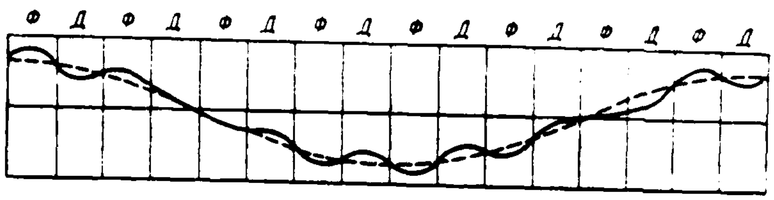

Fig. 2.5.

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The advantages and disadvantages in one or the other type of channel depend on the concrete/specific/actual requirements, part of which will be discussed below. Some possible types of channels are given in Fig. 2.6. The lenses, united structurally/constructurally (for example, arranged/located within one drift tube), are conditionally shown confronting close friend from to friend.

Strong focusing was for the first time proposed for the linear accelerators by Blewett [27] and it is now best of the possible types of focusing. From the moment/torque of the discovery of the principle of strong focusing [38] to the investigation of the rectilinear strong-focusing channels and separate quadrupoles are devoted many works [39-56].

Before passing to the optic/optics of channels, let us examine quadrupole lens. Let us assume for the concrete definition that the lens is magnetic. Then let us transfer results for the electrostatic lens. Without examining thus far the effect of edge effects, let us suppose that lens - infinitely long, so that the problem about field distribution is reduced to the flat/plane: $B_z = 0$. Field in the aperture of lens satisfies the equations of the statics

$$\mathrm{rot}\,\mathbf{B} = 0; \quad \mathrm{div}\,\mathbf{B} = 0.$$

Hence for two-dimensional problem we have

$$\frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x}; \quad \frac{\partial B_x}{\partial x} = -\frac{\partial B_y}{\partial y}; \quad \frac{\partial B_x}{\partial z} = \frac{\partial B_y}{\partial z} = 0. \qquad (2.20)$$

The components of field are changed in the limits of the aperture (see Fig. 2.3).

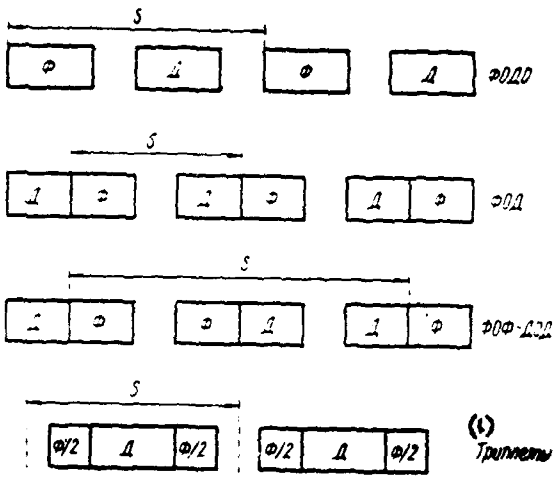

Fig. 2.6.

Key: (1). Triplets.

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For example, component B_x it is positive in the upper half-plane (during the distribution of poles, indicated in Fig. 2.3); with the decrease of ordinate value B_x decreases and then is reversed the sign upon transfer of the ordinate through zero: $B_x<0$ with y<0. It analogously behaves component B_y. Let us introduce into the examination the gradients of the components of the magnetic field

$$\operatorname{grad} B_x = \mathbf{i}\frac{\partial B_x}{\partial x} + \mathbf{j}\frac{\partial B_x}{\partial y};$$
$$\operatorname{grad} B_y = \mathbf{i}\frac{\partial B_y}{\partial x} + \mathbf{j}\frac{\partial B_y}{\partial y}. \quad (2.21)$$

Utilizing equalities (2.20), it is possible to represent the gradient of the vertical component of field in the form

$$\operatorname{grad} B_y = \mathbf{i}\frac{\partial B_x}{\partial y} - \mathbf{j}\frac{\partial B_x}{\partial x}.$$

Consequently,

$$|\operatorname{grad} B_x| = |\operatorname{grad} B_y|; \quad \operatorname{grad} B_x \cdot \operatorname{grad} B_y = 0.$$

Thus, at each point of field the gradients of components are equal on the modulus/module and they are mutually perpendicular. The gradients of components are usually called simply field gradient. Let us introduce designation for the field gradient

$$G = |\operatorname{grad} B_x|. \quad (2.22)$$

The directions of gradient (2.21) are called the median axes of lens. Median axes intersect in the beginning of coordinates always at the right angle.

Let us introduce the potential of the static magnetic field

$$\mathbf{B} = \operatorname{grad} U_\Phi.$$

Potential U_Φ satisfies the equation of Laplace which is conveniently examined to account for quadrupole symmetry in the cylindrical coordinates:

$$\frac{\partial^2 U_\Phi}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial U_\Phi}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 U_\Phi}{\partial \varphi^2} = 0. \quad (2.23)$$

The general solution of equation (2.23), final in zero, takes the *form*

$$U_\Phi(r,\ \varphi)=\sum_{n=1}^{\infty} r^n (a_n \sin n\varphi + b_n \cos n\varphi).$$

where n due to the periodicity of potential along the azimuth - whole real numbers. The quadrupole symmetry of field in the most general case superimposes the following conditions on the potential (see Fig. 2.3):

$$\left.\begin{aligned} U_\Phi(r,\ 0)=U_\Phi\left(r,\ \frac{\pi}{2}\right)=0; \\ U_\Phi(r,\ \varphi)=U_\Phi\left(r,\ \frac{\pi}{2}-\varphi\right). \end{aligned}\right\} \qquad (2.24)$$

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From the first two equalities it follows: $b_n=0;\ n=2k$. From the latter/last condition we obtain k=2m+1. Consequently, the potential of the field, which possesses quadrupole symmetry, contains only the even, through one, harmonics

$$U_\Phi(r,\ \varphi)=\sum_{m=0}^{\infty} a_{2(2m+1)} r^{2(2m+1)} \sin 2(2m+1)\varphi. \qquad (2.25)$$

The first terms of series/row (2.25) exist

$$U_\Phi(r,\ \varphi)=a_2 r^2 \sin 2\varphi + a_6 r^6 \sin 6\varphi + a_{10} r^{10} \sin 10\varphi + \ldots \quad (2.25a)$$

Hence

$$B_\varphi=\frac{1}{r}\cdot\frac{\partial U_\Phi}{\partial\varphi}=2a_2 r\cos 2\varphi+6a_6 r^5\cos 6\varphi+10a_{10} r^9 \cos 10\varphi+\ldots.$$

The vertical component of field at points on the axis of abscissas takes the form

$$B_y(x,\ 0)=B_\varphi(r,\ 0)=2a_2 x+6a_6 x^5+10a_{10} x^9+\ldots \qquad (2.26)$$

Let us name/call the lens of ideal, if field gradient is constant in the section of aperture. The pole of ideal lens they are limited by hyperbolas. Actually/really, with the high accuracy on the surface of the poles, prepared from the magnetic material with large permeability $U_\Phi = \text{const}$. Since x=r cos φ, y=r sin φ,

$$U_\Phi(x,\ y) = 2a_2xy + 2a_6(3x^4 - 10x^2y^2 + 3y^4)\,xy + \ldots.$$

Boundary conditions are satisfied on surfaces of xy=const with $a_6 = a_{10} = \ldots = 0$. In the ideal lens

$$\begin{aligned} U_\Phi &= Gxy; \quad G = \text{const}; \\ B_x &= Gy; \quad B_y = Gx. \end{aligned} \tag{2.27}$$

In the electrostatic lens median axes are turned relative to magnetic lens on 45° (see Fig. 2.3). In expression (2.25a) the sine of dual angle converts/transfers into the cosine. Therefore for the ideal electrostatic lens

$$\begin{aligned} U_\Phi &= \frac{1}{2} G(x^2 - y^2); \\ E_x &= Gx; \quad E_y = -Gy. \end{aligned} \tag{2.28}$$

In the strong-focusing channel, which consists of the ideal lenses, the equations of motion of particles are linear. The nonlinear components of focusing field (2.26) cause the distortions of phase volume, which lead to the losses of particles or a deterioration in focusing [52]. Therefore one should approach that so that the focusing fields would be linear possible.

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Allowance for the nonlinearity of the field of quadrupole lenses in the strong-focusing linear accelerator is evaluated below. In practice it is not possible to perform lens with the ideal hyperbolic poles. In the magnetic lenses it is necessary to leave places for distribution of windings, in the electrostatic ones the length of the generatrix of pole is limited to the permissible breakdown voltage. Meanwhile hyperbolic of pole with ragged generatrix lead in many instances to the nonlinearity of field, which exceeds nonlinearity with the poles of another form. In the small quadrupole lenses, intended for the arrangement/position within drift tubes, usually they prefer to utilize pole with the flat/plane profile/airfoil, which leaves more than winding space. The necessary linearity of focusing field can be ensured, if beam section (taking into account the oscillations/vibrations of beam as whole) it is considerably less than the magnetic aperture of lens. This however, makes it necessary to increase magnetic aperture, which is extremely undesirable from the point of view of maximum core induction and the power scattered in the lens, and sometimes also it is impossible.

From expression (2.26) it is evident that in the nonlinear focusing field are present the members of fifth, the ninth and so forth of degrees. The nonlinearity of field can be considerably decreased, if to fit this form of the rectangular profile/airfoil of the pole piece, with which of expansion (2.26) will fall out the member of fifth degree [53]. Let us note that the suppression of the member of fifth degree is possible not with any form of profile/airfoil. Since the break of hyperbola leads to relative field weakening on the edges of operating region, then for decreasing the nonlinearity of field is necessary the profile/airfoil, which speaks in favor of hyperbola to the side of operating region. One of the possible forms of this profile/airfoil is rectangular pole. Fig. 2.7a gives one of four right-angled pole pieces. For reasons of symmetry it suffices to examine potential distribution in one quadrant of plane x, y.

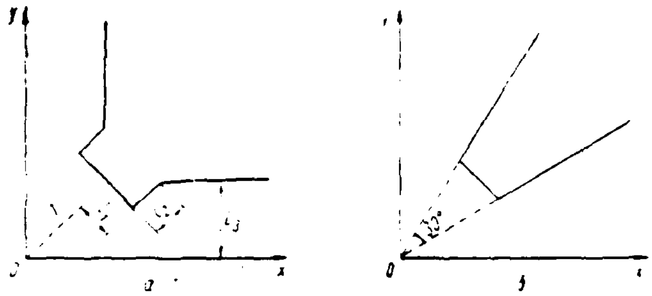

Fig. 2.7.

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If we introduce the complex coordinate z=x+iy, then expression for potential (2.25) will take the form

$$U_{\Phi}(x, y) = \operatorname{Im} \sum_{m=0}^{\infty} a_{2(2m+1)} z^{2(2m+1)}.$$

Hence it is apparent that the series/row

$$W(x, y) = \sum_{m=0}^{\infty} a_{2(2m+1)} z^{2(2m+1)}$$

this is complex potential of the flat/plane quadrupole field

$$W(x, y) = V_{\Phi}(x, y) + iU_{\Phi}(x, y),$$

where V_{Φ} - function of flow. Complex field $B = B_y + iB_x$ is determined by expression [57]

$$B = \frac{dW}{dz} = \frac{\partial U_{\Phi}}{\partial y} + i\,\frac{\partial U_{\Phi}}{\partial x}.$$

Complex potential can be found, if to produce conformal mapping of the region, limited by coordinate semi-axes and the generatrix of pole, onto the single band of plane W then so that the coordinate axes would pass into straight line $U_\Phi = 0$, and generatrix - into straight line $U_\Phi = 1$. According to Schwarz-Christoffel theorem [57] the conversion

$$\frac{dz}{d\zeta} = \frac{C(\zeta^2 - b^2)^{1/2}}{(\zeta^2 - a^2)^{1/4}(\zeta^2 - 1)\zeta^{1/2}}$$

transfers/translates the region of plane z indicated to the upper half-plane of complex variable ζ. The parameters of representation a, b depend on the coordinates of the salient points of profile/airfoil. The constant C is determined by the circuits/bypasses of the infinite point of region on plane z and the point $\zeta = 1$ corresponding to it on the plane ζ. The function, which reflects half-plane to the single band, exists

$$\zeta = \operatorname{th}\frac{\pi W}{2}.$$

Calculation leads to the following expression for the coefficient with the sixth harmonic of the potential

$$a_6 = -\frac{3\pi^4}{128}\cdot\frac{a^2(b^2-1)^2}{b^2(a^2-1)}(2a^2b^2 + 3b^2 - 6a^2),$$

moreover the sizes/dimensions of the pole (see Fig. 2.7a) they are connected with the parameters, a, b with the equalities

$$\frac{l_1}{l_3} = \frac{2}{\pi}\cdot\frac{(a^2-1)^{1/4}}{(b^2-1)^{1/2}}\int\limits_b^\infty \frac{(\zeta^2-b^2)^{1/2}d\zeta}{(\zeta^2-a^2)^{1/4}(\zeta^2-1)\zeta^{1/2}};$$

$$\frac{l_2}{l_3} = \frac{2}{\pi}\cdot\frac{(a^2-1)^{1/4}}{(b^2-1)^{1/2}}\int\limits_a^b \frac{(b^2-\zeta^2)^{1/2}d\zeta}{(\zeta^2-a^2)^{1/4}(\zeta^2-1)\zeta^{1/2}}.$$

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Thus, by the adequate/approaching identification of parameters a, b and respectively the sizes/dimensions of profile/airfoil it is possible to reduce coefficient a_6 to zero. In this case coefficient a_{10} noncritically depends on selection a, b. One of the possible combinations of the sizes/dimensions of the pole piece, which ensures the suppression of the sixth harmonic of potential, exists l_1=0.4852; l_2=0.5741; l_3=0.7700. Distance from the axis to the pole is accepted by the equal to unity. For the comparison Fig. 2.7b shows trapezoidal pole with the apex angle of 30°. This pole is used in the quadrupole lenses of linear accelerator on 50 MeV in CERN. Field measurements showed [56] that on radius 0.75 in the lens with the suppressed sixth harmonic the gradient differs from its value/significance in zero for 5o/o, and in the lens with the pole, shown in Fig. 2.7b, to 20o/o. In the latter/last lens the divergence of gradient of 5o/o occurs on radius 0.57. The relative deflections of field respectively are everywhere less.

In the short lenses occurs the "sag/sagging" of field after the edge of lens and gradient depends on longitudinal coordinate. Potential satisfies the equation

$$\frac{\partial^2 U_\Phi}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial U_\Phi}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 U_\Phi}{\partial \varphi^2}+\frac{\partial^2 U_\Phi}{\partial z^2}=0.$$

Dividing variable/alternating, we obtain the following general/common/total expression for the potential of the short quadrupole lens:

$$U_\Phi(r,\ \varphi,\ z)=\int_{-\infty}^{+\infty}\sum_{m=0}^{\infty} a_{4m+2}(k)\, I_{4m+2}(kr)\sin(4m-2)\varphi e^{ikz}dk. \qquad (2.29)$$

The vertical component of field on the axis of abscissas is equal to

$$B_y(x,\ z)=\int_{-\infty}^{+\infty}\sum_{m=0}^{\infty}(4m+2)\,a_{4m+2}(k)\,\frac{I_{4m+2}(kx)}{x}\,e^{ikz}\,dk.$$

Expanding the modified Bessel functions in power series [28], we obtain

$$B_y(x,\ z)=\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}A_{mj}(z)\,x^{4m+2j+1}. \qquad (2.30)$$

where

$$A_{mj}(z)=\frac{4m+2}{4^{2m+j+1}j!\,(4m+2+j)!}\int_{-\infty}^{+\infty}k^{2(2m+j+1)}a_{4m+2}(k)\,e^{ikz}\,dk.$$

Series/row (2.30) is conveniently represented in the form

$$B_y(x,\ z)=G(z)\,x+\sum_{m=1}^{\infty}A_{m0}(z)\,x^{4m+1}+$$
$$+\sum_{j=1}^{\infty}A_{0j}(z)\,x^{2j+1}+\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}A_{mj}(z)\,x^{4m+2j+1}. \qquad (2.30a)$$

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Function $G(z)=A_{00}(z)$ - field gradient on the axis of the short lens

$$G(z)=\frac{\partial B_y}{\partial x}(0,\ z). \qquad (2.31)$$

Coefficients $A_{0j}(z)$ are determined by the course of gradient on the axis:

$$A_{0j}(z)=\frac{(-1)^j}{2^{2j-1}j!\,(2+j)!}\cdot\frac{d^{2j}G(z)}{dz^{2j}}$$

it is substantially different from zero only in the edge/boundary sections of lens. The remaining coefficients of series/row (2.30a) depend both on the edge effect and on the profile/airfoil of the pole pieces, moreover coefficients $A_{m0}(z)$ during the limitless elongation of lens approach within the limit the appropriate coefficients of series/row (2.26), which corresponds to field with the assigned pole-piece configuration. In contrast to the field of infinitely long lens in resolution (2.30a) are present the members of all odd degrees. However, it is possible to show that in the short lenses the contribution in the refraction of trajectory give only coefficients A_{m0}. Therefore the considerations given above about the selection of pole-piece configuration retain value/significance, also, for the short lenses. It is possible to investigate the effect of nonlinearity on the particle motion in the channel, being limited to the sum

$$B_y(x,\ z)=G(z)\,x+\sum_{m=1}^{n} A_{m0}(z)\,x^{4m+1}. \qquad (2.32)$$

Disregarding all nonlinear terms of expansion, we obtain following approximate field expression in the short lens:

$$B_y(x,\ z)\approx G(z)\,x. \qquad (2.33)$$

If we in the expression for potential (2.29) hold down/retain only the term, quadratic relative to r, then let us arrive at the equality

$$U_\phi(x,\ y,\ z)\approx G(z)\,xy. \qquad (2.34)$$

Hence it follows that approximate equality (2.33) is correct for any values of the ordinates

$$B_y(x,\ y,\ z)\approx G(z)\,x. \qquad (2.33a)$$

Approximation/approach (2.33a) reduces to the linear equations of motion. The simplest approximation of the field of short lens - these are approximation by "square wave": G(z)=const at the length of lens; G(z)=0 out of the lens. This approximation can be refined, if we instead of the real short lens introduce into the examination equivalent lens with the constant gradient and to fit its length and gradient then so that the action of this lens would be in a certain sense equivalent to the action of a real lens. The choice of equivalent lens let us examine below. Fig. 2.8 shows the course of gradient on the axis of real short lens.

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The initial parameters for the rational design of quadrupole lens are the gradient of focusing field and distance from the axis of lens to the pole. The ampere turns, necessary for the creation of field with the assigned gradient in the magnetic lens, can be determined, on the basis of the general/common/total integral relationship/ratio

$$\oint_{\Gamma} \mathbf{H} d\mathbf{l} = \int_{S} \delta \, ds.$$

The convenient way of integration is shown in Fig. 2.9 by dotted line. During the calculation of integral on the left side let us disregard/neglect magnetic intensity in the core. Let us assume a is the distance from the axis to the pole (see Fig. 2.9). Then

$$\oint_{\Gamma} \mathbf{H} d\mathbf{l} = \int_{-a/\sqrt{2}}^{+a/\sqrt{2}} H_y \, dy.$$

In the linear approximation/approach to a field between the poles

$$H_y = \frac{1}{\mu_0} Gx,$$

where $\mu_0 = 4\pi \cdot 10^{-7}$; on the way of integration $x = a\sqrt{2}$. Hence

$$\oint_{\Gamma} \mathbf{Hdl} = \frac{1}{\mu_0} Ga^2.$$

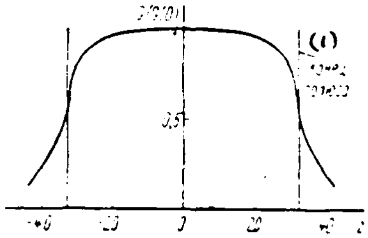

Fig. 2.8.

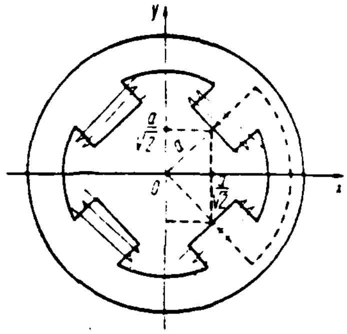

Fig. 2.9.

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Integral in the right side is equal (see Fig. 2.9)

$$\int_S \delta\, ds = 2NI,$$

where NI - number of ampere-turns, which fall to one pole. Equalizing both integrals, we obtain

$$NI = \frac{5}{4\pi} 10^6 \cdot Ga^2.$$

Empiricism established/installed, that to account for steel core a number of ampere-turns should be increased by 10o/o. This gives

$$NI = 0{,}44 \cdot 10^6 Ga^2. \qquad (2.35)$$

In the system of SI units the induction of magnetic field is measured in the tesla: 1 T=10^4 G. Usually during calculations of accelerators is more convenient to measure the induction in the gauss, and the gradient of magnetic field in the gausses per centimeter:

$$G\ \text{гс/см} = 100G^{(1)}\ \text{тл/м}. \qquad (2.36)$$

Key: (1). ml/m.

Converting/transferring in formula (2.35) to the gradient in the gausses per centimeter and to the length in the centimeters, we have

$$NI = 0{,}44Ga^2. \qquad (2.35a)$$

It is easy to obtain also formula for the power, scattered in the quadrupole lens, with the supply of windings by the direct current:

$$P \approx 6{,}1 \cdot \frac{\varrho l}{S_0 f} G^2 a^4. \qquad (2.37)$$

Here P - dissipated power, W; ρ - specific winding impedance, Ω·cm; l - average/mean length of turn, cm; S_0 - total area of windows for positioning/arranging the windings; f - duty factor of window, equal to the ratio of the total cross section of copper to the area of window; G - gradient, G/cm; a - radius of magnetic aperture, cm. From formula (2.37) it is evident that the dissipated power grows/rises proportional to the fourth, the degree of a radius of aperture. This is one of the most serious factors, which limit the aperture of the strong-focusing channel with the supply of quadrupole lenses by direct current.

For the electrostatic lens in linear approximation/approach (2.28) we have

$$V = \pm \frac{1}{2} Ga^2, \qquad (2.38)$$

where G - field gradient, V/cm², V - potential on each electrode relative to the earth/ground (see Fig. 2.3b).

Let us compose the equations of motion of particles in the strong-focusing channel taking into account the space charge of beam and defocusing action of accelerating gaps. Repeating the considerations, given above relative to formula (2.6), we have

$$\frac{d\mathbf{p}}{dt} \approx m_0\gamma \frac{d^2\mathbf{r}}{dt^2}.$$

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Thus,

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{1}{m_0\gamma}\mathbf{F}, \qquad (2.39)$$

where F - total force, which acts on the particle. This force it is possible to represent in the form of three components

$$\mathbf{F} = \mathbf{F}_{\omega} - \mathbf{F}_{\kappa} - \mathbf{F}_{\Phi}.$$

Here F_{ω} - force, which acts on the particle from the side of high-frequency accelerating field. The amount of this force in axisymmetric accelerating field of the type TM was found above [see equality (2.15)]. In the Cartesian coordinates

$$F_{x\omega} = -\frac{e}{2\gamma^2}\cdot\frac{\partial E_z}{\partial z}(z)\,x;$$
$$F_{y\omega} = -\frac{e}{2\gamma^2}\cdot\frac{\partial E_z}{\partial z}(z)\,y; \qquad (2.40)$$
$$F_{z\omega} = eE_z.$$

F_{κ} - the force of electrostatic pushing apart. U, A - scalar and vector potentials of the proper field of the charged/loaded particles in the coordinate system, rigidly connected with the accelerator (laboratory coordinate system). If E_K, B_K - corresponding proper fields, then in quasi-steady-state approximation/approach

$$\mathbf{E}_K = -\operatorname{grad} U; \quad \mathbf{B}_K = \operatorname{rot} \mathbf{A}. \qquad (2.41)$$

Since in the coordinate system, which moves along the longitudinal axis with the particle speed, the vector potential is absent, according to the formulas of the relativistic conversion of potentials, in the laboratory coordinate system we have

$$A_x = A_y = 0; \quad A_z = \frac{v}{c^2} U. \qquad (2.42)$$

Actually/really, let us designate values in the moving/driving coordinate system by indices ~~ts~~ μ; then in general case [58]

$$\left.\begin{aligned} U &= \gamma (U_\text{ц} - vA_{z\text{ц}}); \\ A_x = A_{x\text{ц}};\ A_y &= A_{y\text{ц}};\ A_z = \gamma\left(A_{z\text{ц}} + \frac{\beta}{c} U_\text{ц}\right). \end{aligned}\right\} \qquad (2.43)$$

When $\mathbf{A}_\text{ц} = 0$ we obtain expressions (2.42). Further, in view of the Lorentz decrease of distances $\Delta z_\text{ц} = \gamma \Delta z$, where $\Delta z_\text{ц}$ - interval, measured by observer, who moves together with Ts-system. According to the formulas of the relativistic conversion of fields [58] $E_z = E_{z\text{ц}}$. Hence

$$E_z = -\frac{\partial U_\text{ц}}{\partial z_\text{ц}} = -\frac{1}{\gamma^2} \cdot \frac{\partial U}{\partial z}. \qquad (2.44)$$

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According to expressions (2.41), (2.42), (2.44), the components of the proper field of particles take the form

$$\begin{gathered} E_{x\text{к}} = -\frac{\partial U}{\partial x};\ E_{y\text{к}} = -\frac{\partial U}{\partial y};\ E_{z\text{к}} = -\frac{1}{\gamma^2} \cdot \frac{\partial U}{\partial z}; \\ B_{x\text{к}} = \frac{v}{c^2} \cdot \frac{\partial U}{\partial y};\ B_{y\text{к}} = -\frac{v}{c^2} \cdot \frac{\partial U}{\partial x};\ B_{z\text{к}} = 0. \end{gathered} \qquad (2.45)$$

Substituting expressions (2.45) into the formula for electromagnetic force (2.5) and disregarding the transverse components of speed, we obtain

$$F_{x\kappa}=-\frac{e}{\gamma^2}\cdot\frac{\partial U}{\partial x},\quad F_{y\kappa}=-\frac{e}{\gamma^2}\cdot\frac{\partial U}{\partial y};\quad F_{z\kappa}=-\frac{e}{\gamma^2}\cdot\frac{\partial U}{\partial z} \qquad (2.46)$$

F_{Φ} - the focusing forces, connected with the fields of quadrupole lenses. Substituting the values of fields $B_x(x, y, z)$, $B_y(x, y, z)$ into formula (2.5) and disregarding the transverse components of the speed, we have

$$F_{x\Phi}=-evB_y(x, y, z);\ F_{y\Phi}=evB_x(x, y, z);\ F_{z\Phi}=0. \qquad (2.47)$$

Since $B_x=-\frac{\partial U_\Phi}{\partial x};\ B_y=\frac{\partial U_\Phi}{\partial y}$, that in linear approximation/approach (2.34)

$$F_{x\Phi}=-evG(z)\,x;\ F_{y\Phi}=evG(z)\,y;\ F_{z\Phi}=0. \qquad (2.47a)$$

For electrostatic fields (2.28) in the same linear approximation/approach

$$F_{x\Phi}=eGx;\ F_{y\Phi}=-eGy;\ F_{z\Phi}=0. \qquad (2.48)$$

Hence it is apparent that electrostatic quadrupole is equivalent to magnetic, if field gradient in electrostatic lens $G_э$ is connected with the field gradient of magnetic lens $G_м$ with the relationship/ratio

$$G_э=vG_м. \qquad (2.49)$$

This formula is conveniently represented also in the form

$$G_э\ (в/см^2)=300\beta\cdot G_м^{(1)}\ (гс/см). \qquad (2.49a)$$

Key: (1). G/cm.

The gradients of magnetic quadrupoles in the initial part of the proton linear accelerator have values on the order of 5000 G/cm. Let β=0.04. Then electrostatic is quadrupole, that replaces magnetic, it must have a gradient $G_э = 60\,000$ V/cm². With a radius of aperture a=1 cm this gradient is provided, if the potential of each electrode relative to the earth/ground is V=30 kV, and the stress/voltage between the adjacent electrodes 2V=60 kV. Quadrupole lenses, placed within drift tubes, small sizes/dimensions. The guarantee of the necessary gradient of magnetic field in such lenses does not cause special difficulties. Meanwhile to ensure dielectric strength (on 60 kV) of construction/design within the tube is very difficult. In the accelerator with drift tubes product $\beta G_м$ they usually keep constant.

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Therefore field gradients in the magnetic quadrupoles decrease with an increase in the energy of particles. But the field gradient of electrostatic quadrupoles in this case does not depend from the energy of particles and remains high to the end/lead of the linear accelerator. Therefore electrostatic quadrupoles in the linear accelerators with drift tubes are not utilized. In other constructions/designs electrostatic quadrupoles can prove to be more preferable than magnetic ones, since they have some advantages in comparison with the magnetic quadrupoles. The constructions/designs of electrostatic lenses are simpler, in them to more easily maintain/withstand the necessary form of electrodes, there is no dissipation of power. Furthermore, the electrostatic lenses, placed within the vacuum channels, do not worsen/impair evacuation.

Let us design equation (2.39) on the coordinate axes and will substitute the appropriate values of forces (2.40), (2.46), (2.47). We will obtain the following equations of motion:

$$\frac{d^2x}{dt^2} = -\frac{ev}{m_0\gamma} B_y(x, y, z) - \frac{e}{2m_0\gamma^3} \cdot \frac{\partial E_z}{\partial z}(z, t)\, x - \frac{e}{m_0\gamma^3} \cdot \frac{\partial U}{\partial x};$$
$$\frac{d^2y}{dt^2} = \frac{ev}{m_0\gamma} B_x(x, y, z) - \frac{e}{2m_0\gamma^3} \cdot \frac{\partial E_z}{\partial z}(z, t)\, y - \frac{e}{m_0\gamma^3} \cdot \frac{\partial U}{\partial y}; \quad (2.50)$$
$$\frac{d^2z}{dt^2} = \frac{e}{m_0\gamma} E_z(z, t) - \frac{e}{m_0\gamma^3} \cdot \frac{\partial U}{\partial z}.$$

Latter/last equation - this is the equation of longitudinal vibrations, in detail examined above in variable/alternating ψ, p_ψ and disregarding by the proper field of particles. In the general case in equations (2.50) the variable/alternating are not divided, since potential U depends on three coordinates. In this chapter let us disregard/neglect the collective interactions of particles, after placing $U(x, y, z) \equiv 0$. Since on above (see Chapter I) simplifying assumption accepted the amplitude of accelerating field does not depend on transverse coordinates, latter/last equation does not contain x, y. We examined this equation independent of transverse vibrations of particles. For each particle the equation of longitudinal vibrations has certain solution of z=z(t), which can be substituted in the first two equations. Then the equations of transverse vibrations will not contain longitudinal coordinate. Thus, for the beam with the negligible density of space charge variable/alternating in the equations of motion are divided: the equations of transverse and longitudinal vibrations can be examined independently. Hence, first of all, it follows that in accordance with the theorem. Liouville must independently be retained the values of phase volumes on plane ψ, p_ψ and in four-dimensional space x, y, p_x, p_y.

If we are restricted to linear approximations/approaches to focusing fields, then the equations of transverse vibrations prove to be linear, moreover variable/alternating in them are divided

$$\frac{d^2x}{dt^2} = -\frac{ev}{m_0\gamma} G(t)\,x - \frac{e}{2m_0\gamma^3}\cdot\frac{\partial E_z}{\partial z}(t)\,x;$$
$$\frac{d^2y}{dt^2} = \frac{ev}{m_0\gamma} G(t)\,y - \frac{e}{2m_0\gamma^3}\cdot\frac{\partial E_z}{\partial z}(t)\,y. \quad (2.51)$$

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In this case must remain invariant the phase volume for each of the phase planes x, p_x and y, p_y.

The coefficients of equations (2.51) depend on time. Coefficient $\frac{\partial E_z}{\partial z}$ is periodic with the period of the accelerating structure L. Coefficient G is periodical with the period of focusing field S. Usually the period of focusing field contains the integer of periods of the accelerating structure. Subsequently we will consider that the relationship/ratio of periods indicated is fulfilled. Then the equations of transverse vibrations (2.51) are reduced to the linear equations with the periodic coefficients. It is convenient to switch over to the dimensionless variable τ, with which the period of focusing field is equal to unity,

$$d\tau = \frac{v_s}{S}\, dt. \qquad (2.52)$$

Producing the replacement of variable/alternating in equations (2.51), let us rewrite them in the form

$$\begin{aligned} &\frac{d^2x}{d\tau^2} - Q_x(\tau)\, x = 0; \\ &\frac{d^2y}{d\tau^2} - Q_y(\tau)\, y = 0. \end{aligned} \qquad (2.53)$$

Functions Q_x, Q_y satisfy periodicity condition

$$Q(\tau - 1) = Q(\tau). \qquad (2.54)$$

Into equations (2.53) are introduced the designations

$$\begin{aligned} &Q_x(\tau) = Q_\Phi(\tau) - Q_\omega(\tau); \\ &Q_y(\tau) = -Q_\Phi(\tau) - Q_\omega(\tau). \end{aligned} \qquad (2.55)$$

where Q_Φ - function, determined by focusing fields. Disregarding the scatter of longitudinal velocities, we have

$$Q_\Phi(\tau) = S^2 \frac{eG(\tau)}{p_s}, \qquad (2.56)$$

where p_s - the longitudinal component of the impulse/momentum/pulse of synchronous particle; Q_ω - function, which describes the defocusing action of accelerating field

$$Q_\omega(\tau) = \frac{eS^2}{2\mathcal{E}_0\beta^2\gamma^3} \cdot \frac{\partial E_z}{\partial z}(\tau). \qquad (2.57)$$

Before passing to the analysis of equations (2.53), let us examine the focusing action of short lenses. According to equation (2.50), the particle motion, which lies at plane XOZ, in the field of lens, free from accelerating fields and space charge, is described by the equation

$$\frac{d^2x}{dz^2} = -\frac{e}{p} B_y(x, z). \qquad (2.58)$$

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Refraction of trajectory in the lens is equal

$$\Delta \frac{dx}{dz} = -\frac{e}{p} \int_{-\infty}^{+\infty} B_y(x, z)\, dz. \qquad (2.59)$$

Let us name/call the lens of thin, if it is possible to consider that at the length of lens the particle displacement does not manage to change. The focusing action of thin lens is completely determined by refraction of trajectory (2.59). Substituting series/row (2.30) into integral (2.59), for the thin lens we obtain

$$\Delta \frac{dx}{dz} = -\frac{e}{p} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} x^{4m+2j+1} \int_{-\infty}^{+\infty} A_{mj}(z)\, dz. \qquad (2.60)$$

Let us examine the integrals, entering sum (2.60). $B_{mj}(k)$ - function, conjugated/combined with function $A_{mj}(z)$:

$$A_{mj}(z) = \int_{-\infty}^{+\infty} B_{mj}(k)\, e^{ikz}\, dk;$$

$$B_{mj}(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} A_{mj}(z)\, e^{-ikz}\, dz.$$

Then

$$\int_{-\infty}^{+\infty} A_{mj}(z)\,dz = B_{mj}(0).$$

From the expressions for coefficients A_{mj} [see expression (2.30) and the following after it] can be obtained

$$B_{mj}(k) \doteqdot k^{2(2m+j+1)} a_{4m+2}(k). \tag{2.61}$$

Since refraction of trajectory (2.60) is limited, from relationship/ratio (2.61) it follows that coefficient $B_{m0}(0)$, proportional

$$B_{m0}(0) \doteqdot \lim_{k\to 0} k^{4m+2} a_{4m+2}(k)$$

is a value final. But in this case with $j \neq 0$

$$B_{mj}(0) \doteqdot \lim_{k\to 0} k^{4m+2} k^{2j} \cdot a_{4m+2}(k) = 0.$$

Thus, the contribution to sum (2.60) give only the terms, which contain coefficients $A_{m0}(z)$. Page 78.

§ 2.3. Equations of Mathieu-Hill and function of Floquet.

Particle trajectories in the strong-focusing channel are the solutions of equations (2.53). Linear equations of such type with the periodic coefficients are called the equations of Mathieu-Hill [59]. The equations of Mathieu-Hill are obtained during the analysis of particle motion in any linear periodic structures. In particular, by the equations of Mathieu-Hill are described particle trajectories not only in the strong-focusing channels, but also in any long channels, which consist of the set of discrete/digital lenses. The investigation of the parametric resonances, connected with periodic modulation of the parameters of dynamic system, also always reduces to these equations. Since any accelerators, including linear, are in this or another form periodic structures, equations of type (2.53) have fundamental value/significance in the theory of accelerators. Therefore it is considered advisable for simplification in the following presentation to stop at the character of the solutions of the equations of Mathieu-Hill.

In equations (2.53) the variable/alternating are divided. Let us examine one of these equations, after dropping/omitting the indices

$$\frac{d^2x}{d\tau^2} - Q(\tau)\,x = 0. \tag{2.62}$$

First let us assume that $Q(\tau)$ - the arbitrary function of time. $u(\tau)$, $v(\tau)$ - two solutions of equation (2.62), which correspond to different initial conditions. The Wronskian (Wronskian determinant) pairs of the solutions of the linear equation of the second order, which does not contain first-order derivative, there is, value, not depending on the time:

$$W = \begin{vmatrix} u(\tau) & v(\tau) \\ \frac{du}{d\tau}(\tau) & \frac{dv}{d\tau}(\tau) \end{vmatrix} = \text{const.} \tag{2.63}$$

So that two arbitrary solutions would be linearly independent, it is necessary and sufficient so that the Wronskian determinant of these solutions would differ from zero. Let us accept for future reference the following condition for the standardization of the real fundamental pair of the solutions:

$$u\frac{dv}{d\tau}-v\frac{du}{d\tau}=1. \qquad (2.64)$$

In particular, this condition for standardization satisfies the pair of solutions with the initial conditions

$$u(0)=1;\ \frac{du}{d\tau}(0)=0;\quad v(0)=0;\ \frac{dv}{d\tau}(0)=1.$$

Let us take as the fundamental pair two linearly independent complex solutions

$$\chi_1(\tau)=u_1(\tau)+iv_1(\tau);$$
$$\chi_2(\tau)=u_2(\tau)+iv_2(\tau).$$

It is expressed the second pair of the linearly independent real solutions through the first

$$u_2=a_{11}u_1+a_{12}v_1;$$
$$v_2=a_{21}u_1+a_{22}v_1.$$

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Composing the Wronskian determinant of complex solutions, we obtain

$$W=\begin{vmatrix}\chi_1 & \chi_2\\ \frac{d\chi_1}{d\tau} & \frac{d\chi_2}{d\tau}\end{vmatrix}=$$
$$=[a_{12}+a_{21}+i(a_{22}-a_{11})]\cdot\begin{vmatrix}u_1 & v_1\\ \frac{du_1}{d\tau} & \frac{dv_1}{d\tau}\end{vmatrix}.$$

Since functions χ_1, χ_2 are linearly independent, binomials $a_{12}+a_{21}$ and $a_{22}-a_{11}$ simultaneously do not become zero. If χ_1, χ_2 are selected complex conjugate, then $a_{12}=a_{21}=0$; $a_{11}=-a_{22}=1$. Hence, under the condition standardization for real fundamental pairs (2.64), for the complex conjugate solutions we have

$$\chi \frac{d\chi^*}{d\tau} - \chi^* \frac{d\chi}{d\tau} = -2i. \qquad (2.65)$$

Let us represent the complex conjugate fundamental pair of solutions in the exponential form

$$\chi(\tau) = \sigma(\tau) e^{i\psi(\tau)}; \quad \chi^*(\tau) = \sigma(\tau) e^{-i\psi(\tau)}. \qquad (2.66)$$

Then any real solution takes the form

$$x(\tau) = a\chi(\tau) + a^*\chi^*(\tau) = A\sigma(\tau)\cos[\psi(\tau) - \Theta], \qquad (2.67)$$

where A, Θ - arbitrary real constants, which depend on the initial conditions

$$a = \frac{1}{2} A e^{i\Theta}.$$

Substituting expressions (2.66) into the condition for standardization (2.65), we obtain following common connection/communication between rate of change in the phase of solution and modulus/module of the solution:

$$\frac{d\psi}{d\tau} = \frac{1}{\sigma^2}. \qquad (2.68)$$

Further, if we substitute solution in the form (2.66) into initial equation (2.62) and to consider dependence (2.68), then we will obtain the equation which satisfies the modulus/module of any complex solution:

$$\frac{d^2\sigma}{d\tau^2} + Q(\tau)\sigma - \frac{1}{\sigma^3} = 0. \qquad (2.69)$$

The general/common/total properties of the solutions of the linear second order equation indicated with the coefficient, which are the arbitrary function of time, play important role during the calculation of the focusing of intense beams.

Let us assume now that function $Q(\tau)$ - is periodic (2.54). In principle any pair of the linearly independent solutions of the equation of Mathieu-Hill can be selected as the fundamental.

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Remaining solutions are the linear combination of the chosen characteristic functions. However, not any selection of characteristic functions is equally convenient. It is possible to show that there is always such pair of fundamental solutions of the equation of Mathieu-Hill, through which all remaining solutions are expressed by the simplest form. For the explanation let us examine the trivial case when the coefficient of equation (2.62) is constant:

$$\frac{d^2x}{d\tau^2}+\omega^2x=0. \qquad (2.70)$$

As the complex conjugate pair of the solutions of this equation are always taken the functions

$$\chi(\tau)=e^{i\omega\tau}; \quad \chi^*(\tau)=e^{-i\omega\tau}.$$

Then real solution is represented in the form

$$x(\tau)=A\cos(\omega\tau+\Theta), \qquad (2.71)$$

from which evident that any solution of equation (2.70) - harmonic function with the constant amplitude and the phase, which linearly depends on the time. The normalized characteristic functions take the form

$$\chi = \frac{1}{\sqrt{\omega}} e^{i\omega\tau}; \quad \chi^* = \frac{1}{\sqrt{\omega}} e^{-i\omega\tau}.$$

Substantially the less successful selection of characteristic functions would be, for example, such

$$\chi(\tau) = \frac{2}{\sqrt{3\omega}} \left(e^{i\omega\tau} - \frac{1}{2} e^{-i\omega\tau} \right);$$
$$\chi^*(\tau) = \frac{2}{\sqrt{3\omega}} \left(e^{-i\omega\tau} - \frac{1}{2} e^{i\omega\tau} \right).$$

In this case, as it is easy to show, real solutions (2.67) take the form

$$x(\tau) = \frac{A}{\sqrt{3\omega}} \sqrt{1 - 8\cos^2 \omega\tau} \cos[\psi(\tau) + \Theta]. \quad (2.72)$$

moreover

$$\frac{d\psi}{d\tau} = \frac{3\omega}{1 + 8\cos^2 \omega\tau}.$$

Amplitude and phase of solution (2.72) prove to be the complex functions of time, moreover to be dismantled/selected at the fact that function (2.72) coincides with function (2.71), is difficult. If we deal concerning the equation of harmonic oscillator (2.70), then the amplitude of solution only then is constant, when the phase of solution - linear function of time. However, for the equation of Mathieu-Hill in the general case the selection of the adequate/approaching pair of fundamental solutions is not so trivial.

Let us examine the arbitrary linearly independent pair of the complex solutions of the equation of Mathieu-Hill $\chi_1(\tau)$ and $\chi_2(\tau)$.

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In view of the invariance of equation (2.62) relative to the conversion of the independent variable $\tau'=\tau+1$, in any assigned interval of function $\chi_1(\tau-1)$, $\chi_2(\tau+1)$ they will also be the solutions of equation (2.62):

$$\begin{aligned}\chi_1(\tau-1) &= a_{11}\chi_1(\tau)+a_{12}\chi_2(\tau);\\ \chi_2(\tau+1) &= a_{21}\chi_1(\tau)+a_{22}\chi_2(\tau),\end{aligned}\qquad(2.73)$$

where a_{ij} - constant complex numbers corresponding to this, arbitrarily selected fundamental pair of complex functions. We form certain new complex solution of equation (2.62)

$$\varphi(\tau)=A\chi_1(\tau)+B\chi_2(\tau).$$

The function

$$\varphi(\tau-1)=A\chi_1(\tau-1)+B\chi_2(\tau-1)$$

taking into account conversion (2.73) can be represented in the form

$$\varphi(\tau+1)=(Aa_{11}+Ba_{21})\chi_1(\tau)+(Aa_{12}-Ba_{22})\chi_2(\tau).$$

Let us select conversion factors A, B so that would be satisfied the identity

$$\varphi(\tau+1)=\lambda\varphi(\tau),\qquad(2.74)$$

where λ - a constant value. condition (2.74) is satisfied, if conversion factors satisfy the equations

$$\begin{aligned}Aa_{11}+Ba_{21}&=\lambda A;\\ Aa_{12}+Ba_{22}&=\lambda B.\end{aligned}\qquad(2.75)$$

The system of linear homogeneous equations (2.75) has nontrivial solutions with

$$\begin{vmatrix} a_{11}-\lambda & a_{21} \\ a_{12} & a_{22}-\lambda \end{vmatrix} = 0.$$

Thus, λ - this is the square root equation

$$\lambda^2-(a_{11}-a_{22})\lambda-(a_{11}a_{22}-a_{12}a_{21}) - 0. \qquad (2.76)$$

Let us show that the combinations of values $a_{..}$ forming the coefficients of quadratic equation (2.76), do not depend on the selection of fundamental solutions χ_1, χ_2. Let us introduce for the decrease of recordings the following designations for the columns and the matrices/dies

$$(\chi)_\tau = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix};$$
$$(a) = \begin{pmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{pmatrix}; \qquad (2.77)$$
$$(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

end section.

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The determinant of matrix/die (a) let us designate (a). Then linear transformation (2.73) is written in the matrix form in the form

$$(\chi)_{\tau+1} = (a)(\chi)_{\tau}. \qquad (2.73a)$$

while equation (2.76) is reduced to the form

$$|(a) - \lambda(E)| = 0. \qquad (2.76a)$$

Let us examine now certain other pair of complex fundamental solutions $\varkappa_1(\tau)$, $\varkappa_2(\tau)$.

We have

$$(\varkappa)_{\tau+1} = (b)(\varkappa)_{\tau}.$$

New solutions can be expressed through the old ones

$$(\varkappa)_{\tau} = (c)(\chi)_{\tau}.$$

By hence consecutive substitution we obtain

$$(\varkappa)_{\tau+1} = (c)(a)(c)^{-1}(\varkappa)_{\tau},$$

where $(c)^{-1}$ - matrix/die, to reciprocal matrix (c):

$$(c)(c)^{-1}=(E).$$

Thus,

$$(b)=(c)(a)(c)^{-1}. \qquad (2.78)$$

If we as the fundamental solutions select x_1 and x_2, the equation for λ would take the form

$$|(b)-\lambda(E)|=0.$$

Let us substitute into the latter/last equality expression for the matrix/die (b) (2.78). Taking into account the identity

$$(E)=(c)(E)(c)^{-1}$$

we will obtain

$$|(b)-\lambda(E)|=|(c)\{(a)-\lambda(E)\}(c)^{-1}|.$$

Since the determinant of matrix product is equal to the product of the determinants of these matrices/dies, then

$$|(b)-\lambda(E)|\equiv|(a)-\lambda(E)|. \qquad (2.79)$$

Both polynomials (2.79) have identical roots λ_1, λ_2 and, therefore, the coefficients of these polynomials coincide:

$$-(\lambda_1+\lambda_2)=a_{11}+a_{22}=b_{11}+b_{22};$$
$$\lambda_1\lambda_2=|(a)|=|(b)|.$$

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Equation (2.76) does not depend on the selection of characteristic functions and it is uniquely determined by initial differential equation (2.62). To each root of characteristic equation (2.76) corresponds its, specific for this pair of characteristic functions, ratio A/B. We will obtain two functions, which satisfy condition (2.74):

$$\begin{aligned} \varphi_1(\tau) &= A_1\chi_1(\tau) + B_1\chi_2(\tau); \\ \varphi_2(\tau) &= A_2\chi_1(\tau) + B_2\chi_2(\tau). \end{aligned} \qquad (2.80)$$

moreover

$$\varphi_1(\tau+1) = \lambda_1\varphi_1(\tau); \quad \varphi_2(\tau+1) = \lambda_2\varphi_2(\tau). \qquad (2.81)$$

According to expression (2.75),

$$\frac{B_1}{A_1} = \frac{\lambda_1 - a_{11}}{a_{21}}; \quad \frac{A_2}{B_2} = \frac{\lambda_2 - a_{22}}{a_{12}}. \qquad (2.82)$$

In this case it is assumed that characteristic equation (2.76) does not have multiple roots. By equalities (2.80) of function $\varphi_1(\tau)$, $\varphi_2(\tau)$ they are determined with an accuracy to arbitrary complex factors. Let us compose the Wronskian determinant of these functions. Utilizing conversion (2.80), we obtain

$$\begin{vmatrix} \varphi_1 & \varphi_2 \\ \frac{d\varphi_1}{d\tau} & \frac{d\varphi_2}{d\tau} \end{vmatrix} = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \cdot \begin{vmatrix} \chi_1 & \chi_2 \\ \frac{d\chi_1}{d\tau} & \frac{d\chi_2}{d\tau} \end{vmatrix}.$$

Let us substitute into the determinant, comprised of the conversion factors, relations (2.82). This it gives

$$A_1B_2-A_2B_1=\frac{A_1A_2}{a_{21}}(\lambda_2-\lambda_1). \qquad (2.83)$$

If none of the functions φ_1, φ_2 is equal identically to zero ($A_1 \neq 0$, $A_2 \neq 0$), then determinants (2.83) it is different from zero. Functions φ_1, φ_2 are linearly independent. It is obvious that other functions, satisfying the assigned equation of Mathieu-Hill, to condition (2.74), and linearly independent from φ_1, φ_2, there does not exist. Actually/really, the existence of the third function contradicts so that the system of linear equations (2.75) has only two series/rows of the nontrivial solutions A/B.

The absolute term of characteristic equation (2.76) is determined from the condition of the constancy of Wronskian determinant. $W(\tau)$, $W(\tau+1)$ - the Wronskian determinants of functions χ_1, χ_2 at the moments of time τ and $\tau+1$. Let us substitute into determinant $W(\tau+1)$ expressions (2.73). It is easy to ascertain that

$$W(\tau+1)=|a|\,W(\tau).$$

Since $W(\tau)$=const,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=1. \qquad (2.84)$$

On the assigned equation of Mathieu-Hill (2.62) depends only the coefficient of characteristic equation with the first degree λ. Let us introduce the following designation for this coefficient

$$a_{11}+a_{22}=2\,\mathrm{ch}\,l. \qquad (2.85)$$

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The sum of diagonal matrix elements is called trace, or bore-hole, this matrix/die

$$\mathrm{ch}\,l=\frac{1}{2}\,Sp\,(a). \qquad (2.85a)$$

Characteristic equation (2.76) is simplified

$$\lambda^2-2\lambda\,\mathrm{ch}\,l+1=0. \qquad (2.86)$$

Since the coefficients of characteristic equation do not depend on the selection of characteristic functions, it is possible, without disrupting generality, to select the complex conjugate pair of solutions $\chi_1(\tau)=\chi(\tau);\ \chi_2(\tau)=\chi^*(\tau)$. If

$$\chi(\tau+1)=a_{11}\chi(\tau)+a_{12}\chi^*(\tau),$$

then, obviously,

$$\chi^{*}(\tau+1)=a_{12}^{*}\chi(\tau)+a_{11}^{*}\chi^{*}(\tau).$$

In this case $a_{22}=a_{11}^{*}$

and

$$\operatorname{ch} l=\frac{1}{2}(a_{11}+a_{11}^{*}).$$

Value ch$l$ as the sum of two complex conjugate numbers, is real. Consequently, the roots of characteristic equation can be only real or only complex conjugate.

According to expression (2.86), $\lambda_1\lambda_2=1$,

or

$$\lambda_1=\lambda;\quad \lambda_2=\frac{1}{\lambda}. \tag{2.87}$$

Thus, there is an only pair of the linearly independent solutions of the equation of Mathieu-Hill, which satisfies the conditions

$$\varphi_1(\tau+1)=\lambda\varphi_1(\tau);\quad \varphi_2(\tau+1)=\frac{1}{\lambda}\varphi_2(\tau). \tag{2.88}$$

This confirmation is called the Floquet theorem. The fundamental pair of the solutions of the equation of Mathieu-Hill, which satisfies conditions (2.88), is called Floquet's functions. Floquet's functions exactly are that chosen pair of the fundamental solutions, with the aid of which most simply are expressed any other solutions of the equation of Mathieu-Hill. In particular, functions $e^{\pm i\omega\tau}$ are to Floquet's function the equation of harmonic oscillator (2.70). However, the value/significance of Floquet's functions for the channels, which have periodic structure, is much wider. Subsequently let us show that with the optimum focusing envelope of particles of particles in the long channel is described by Floquet functions of this channel.

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Floquet's functions in the general case are complex. If moduli/modules of both Floquet's functions are limited in the time, then the general solution of the equation of Mathieu-Hill, since it is the linear combination of Floquet's functions, it is stable. But if the modulus/module at least of one function of Floquet unlimitedly grows/rises in the course of time, then the general solution is unstable. From conditions (2.88) it follows

$$\varphi_1(\tau-1) = \lambda\ \varphi_1(\tau),$$
$$\varphi_2(\tau-1) = \frac{1}{\lambda}\ \varphi_2(\tau).$$

Hence it is apparent that the general solution is always stable, if $\lambda = 1$. But if $\lambda \neq 1$, then one of the moduli/modules unlimitedly grows/rises and the general solution is unstable. According to the condition the cases of multiple roots let us exclude. Let the roots of characteristic equation be real; then $|\lambda|\neq 1$ and the general solution is unstable. When the roots of characteristic equation are complex conjugated, then, according to expression (2.87), $|\lambda|=1$ and the general solution is stable.

The roots of characteristic equation (2.86) take the form

$$\lambda = \mathrm{ch}\, l \pm \sqrt{\mathrm{ch}^2 l - 1} = e^{\pm l}. \qquad (2.89)$$

Parameter l is called the characteristic index of the equation of Mathieu-Hill. Let

$$l = k + i\mu. \qquad (2.90)$$

Then

$$\mathrm{ch}\,(k + i\mu) = \mathrm{ch}\, k \cos\mu - i\, \mathrm{sh}\, k \sin\mu.$$

Since $\mathrm{ch}\, l$ - value real, are possible only the two cases.

1) $u = \pm n\pi$; $l = k \pm in\pi$, where n - any integers.

In the case

$$\lambda_1 = (-1)^n e^k; \quad \lambda_2 = (-1)^n e^{-k}.$$

The general solution is unstable: $|\lambda|\neq 1$. Depending on the values of the parameters, which are determining function $Q(\tau)$, the equation has an infinite set of unstable regions to each of which corresponds the specific value/significance n. To real roots it corresponds

$$\operatorname{ch} l > 1.$$

2) $k=0;\quad l=i\mu.$

The general solution is stable: $\lambda=e^{\pm i\mu}$; $\lambda=1$. The parameter μ can be represented in the form $\mu=\tilde{\mu}\pm n\pi$, where $0<\tilde{\mu}<\pi$. Then

$$\lambda_1=(-1)^n e^{i\tilde{\mu}};\quad \lambda_2=(-1)^n e^{-i\tilde{\mu}}.$$

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To each value/significance n corresponds the specific range of change in the parameters of function $Q(\tau)$, for which the general solution is stable. Stability condition is the inequality

$$|\operatorname{ch} l|<1. \qquad (2.91)$$

When both cases occur simultaneously:

$$l=\pm in\pi,$$

then $\operatorname{ch} l=\pm 1$ and characteristic equation (2.86) possesses multiple root. The values of characteristic index indicated answer stability limits.

Let us present Floquet's function as follows

$$\varphi_1(\tau) = e^{l\tau}F_1(\tau); \quad \varphi_2(\tau) = e^{-l\tau}F_2(\tau). \tag{2.92}$$

We have

$$\varphi_1(\tau+1) = e^{l(\tau+1)}F_1(\tau+1).$$

At the same time, according to expression (2.88),

$$\varphi_1(\tau+1) = \lambda\varphi_1(\tau) = e^{l}e^{l\tau}F_1(\tau).$$

Hence

$$F_1(\tau+1) = F_1(\tau).$$

Of functions F_1, F_2 - are periodic, with the period $\Delta\tau=1$, i.e., with the period of function $Q(\tau)$.

Let us examine the solutions of the equation of Mathieu-Hill in the unstable regions. Substituting in equalities (2.92) expression for l (2.90), we obtain

$$\varphi_1(\tau) = e^{h\tau}U_1(\tau); \quad \varphi_2(\tau) = e^{-h\tau}U_2(\tau), \tag{2.93}$$

where

$$\begin{aligned} U_1(\tau) &= e^{\pm in\pi\tau}F_1(\tau); \\ U_2(\tau) &= e^{\mp in\pi\tau}F_2(\tau). \end{aligned} \tag{2.94}$$

Functions (2.94) - periodic. The frequency of these functions with even n coincides with the period of a function $Q(\tau)$, and with odd n it is twice lower. Actually/really, cyclic period of a function $F_1(\tau)$, $F_2(\tau)$ is 2π; cyclic period of a function $e^{in\pi\tau}$ is equal to $n\pi$. Therefore with even n in the period $\Delta\tau=1$ is placed the integer of periods of function $e^{in\pi\tau}$. With odd n the integer of periods of function $e^{in\pi\tau}$ is placed only in the interval $\Delta\tau=2$. As it is possible to show, in the unstable region always it is possible to select the arbitrary complex factors of conversion (2.80) so that Floquet's functions (2.93) would be the real functions of time. Let us introduce instead of functions (2.94) other two real functions with the aid of the relationships/ratios:

$$U_1(\tau)=\varrho(\tau)\cos\Phi(\tau);$$
$$U_2(\tau)=\varrho(\tau)\sin\Phi(\tau).$$

Then the general real solution in the unstable region will take the form

$$x(\tau)=\varrho(\tau)[Ae^{h\tau}\cos\Phi(\tau)+Be^{-h\tau}\sin\Phi(\tau)]. \qquad (2.95)$$

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If initial conditions correspond A=0, then particular solution is stable. In all remaining cases particular solutions are unstable. Function $\rho(\tau)$ is periodic with the period $\Delta\tau=1$ or $\Delta\tau=2$ depending on parity n. Phase change F in the period of solution $\Delta\tau=1$ or $\Delta\tau=2$ composes integer 2π. The increment of oscillation buildup of the unstable part of solution (2.95) is a constant value and is determined by the characteristic index of the equation of Mathieu-Hill. Solutions in the unstable region are important during the investigation of parametric resonances.

During the analysis of transverse vibrations basic interest they will present solutions in the stable regions, or, is more precise, in the first stability region, which corresponds n=0. In the stable region Floquet's functions always can be selected complex conjugate. In fact, let $\chi_1=\chi; \chi_2=\chi^*$. In this case $a_{21}=a_{12}^*$. Utilizing expression (2.82) instead of general/common/total conversion (2.80), we obtain

$$\varphi_1=A_1\left(\chi+\frac{\lambda_1-a_{11}}{a_{12}^*}\chi^*\right);$$
$$\varphi_2=B_2\left(\chi^*+\frac{\lambda_2-a_{11}^*}{a_{12}}\chi\right).$$

In the stable region $\lambda_2=\lambda_1^*$, so that bracketed expressions are complex conjugated. Selecting $B_2=A_1^*$, we have

$$\varphi_1(\tau)=\varrho(\tau)\,e^{i\Phi(\tau)};\quad \varphi_2(\tau)=\varrho(\tau)\,e^{-i\Phi(\tau)}. \qquad (2.96)$$

In accordance with the Floquet theorem

$$\varrho(\tau+1)\,e^{i\Phi(\tau+1)}=e^{i\mu}\varrho(\tau)\,e^{i\Phi(\tau)}.$$

Hence

$$\varrho(\tau+1)=\varrho(\tau);$$
$$\Phi(\tau+1)-\Phi(\tau)=\mu. \qquad (2.97)$$

The modulus/module of Floquet's function in the stable region - this is the periodic function of time with the period $\Delta\tau=1$. The phase of Floquet's function in each period $\Delta\tau=1$ grows/rises by the constant value μ, which does not depend on the reference point of period. Any real solution of the equation of Mathieu-Hill in the stable region can be expressed through phase and modulus/module of Floquet's functions

$$x(\tau)=A\varrho(\tau)\cos[\Phi(\tau)+\Theta]. \qquad (2.98)$$

Thus, if the phase of solution is Floquet's phase, then the amplitude of solution - periodic function of time with the period of a change in the coefficient of equation. Equalities (2.95), (2.98) - the simplest expressions for the general solutions of the equation of Mathieu-Hill. The dependence of modulus/module and phase of Floquet on the time is given in Fig. 2.10.

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According to the general condition for the standardization of complex conjugate solutions (2.65),

$$\varphi\frac{d\varphi^*}{d\tau}-\varphi^*\frac{d\varphi}{d\tau}=-2i. \qquad (2.99)$$

Floquet's functions are completely determined, if is calculated the modulus/module of function, since in accordance with expression (2.68)

$$\Phi(\tau)=\int_0^{\tau}\frac{d\tau}{\varrho^2}. \qquad (2.100)$$

Since the modulus/module of any complex solution satisfies equation (2.69), then the modulus/module of Floquet's function is the periodic solution of this equation. Is given below the method, which makes it possible in certain cases to immediately find initial conditions for the periodic solution. But if initial conditions for the periodic solution cannot be determined sufficiently simply, then equation (2.69) is necessary to decide in electronic computer the method of successive approximations, attaining the coincidence of values of modulus/module and its first-order derivative at both ends/leads of the period.

Expression (2.98) describes, in particular, the trajectory of individual particle in the strong-focusing channel. In this case the parameter μ - phase change of transverse vibrations in one period of focusing field. The instantaneous frequency of transverse vibrations let us name/call rate of change in Floquet's phase:

$$\omega_r=\frac{d\Phi}{dt}. \qquad (2.101)$$

Let T_Φ— time, for which the particle flies one period of the focusing field:

$$T_\Phi = \frac{S}{v_s}. \quad (2.102)$$

Then

$$\omega_r = \frac{1}{T_\Phi Q^2}. \quad (2.103)$$

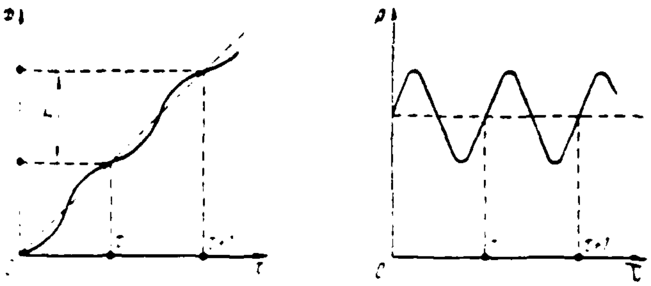

Fig. 2.10.

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Let us introduce also the concept of the medium frequency of transverse vibrations in the period

$$\Omega_r = \frac{1}{T_\Phi} \int_t^{t+T_\Phi} \omega_r(t)\,dt. \qquad (2.104)$$

When $T_\Phi = $ const instantaneous frequency proves to be the periodic function of time, and medium frequency - by a constant value. According to expressions (2.97), (2.100),

$$\mu = \int_\tau^{\tau+1} \frac{d\tau}{\rho^2}. \qquad (2.105)$$

Hence

$$\Omega_r = \frac{\mu}{T_\Phi}. \qquad (2.106)$$

From equality (2.106) it is evident that the parameter μ can be also defined as the average (cyclic) frequency of transverse vibrations to scale of time τ.

Let us fix the moment/torque of time τ_0 and will examine discrete/digital points to the trajectories, distant behind each other to the integer of periods. After the n periods

$$\varrho(\tau_0 - n) = \varrho(\tau_0);$$
$$\Phi(\tau_0 + n) = \Phi(\tau_0) - \mu n.$$

Discrete/digital points lie/rest on the sinusoid

$$x(\tau_0 + n) = A\varrho(\tau_0)\cos[\mu n - \Phi(\tau_0) - \theta], \qquad (2.107)$$

frequency of which is equal to the medium frequency of transverse vibrations. The amplitude of sinusoid in expression (2.107) depends on the chosen phase of period τ_0. Amplitude has the greatest value/significance, if point τ_0 corresponds to the maximum of Floquet's modulus/module, and the smallest value/significance, if τ_0 it answers the minimum. Particle trajectory is placed between these two extreme sinusoids. From the qualitative considerations about the work of quadrupole lenses it is obvious that the maximum of Floquet's modulus/module falls in the focusing section of period, and the minimum - to that defocusing. The parameters of the strong-focusing channel are usually selected by such that the stability of trajectories would be provided in first region (n=0). In this case the medium frequency of transverse vibrations considerably lower than repetition frequency of the focusing structure and particle trajectory is the "slow" sinusoid, to which are superimposed "rapid" oscillations with the period of the focusing field (see Fig. 2.5).

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Some goals, connected with the parametric excitation of forced oscillations, by modulation of frequency, by the oscillations of the membranes/diaphragms, and others can be brought under the specific simplifying assumptions to the particular form of the equation of Mathieu-Hill

$$\frac{d^2x}{d\tau^2}+\pi^2(a-2q\sin 2\pi\tau)x=0, \qquad (2.108)$$

to called equation Mathieu. The coefficient of the equation of Mathieu is the harmonic function of the independent variable. The form of writing of the equation of Mathieu (2.108) is selected here in such a way as to retain the designations of coefficients, accepted in book [59]. Equation (2.108) describes also particle trajectories in the strong-focusing channel, if the gradients of focusing field are distributed along the axis according to sinusoidal law [60]. The solutions of the equation of Mathieu are studied most fully. Fig. 2.11 shows the regions, which correspond to the solutions of the equation of Mathieu with different character of stability [59]. The regions of stable solutions are shaded. In the unstable regions dotted line noted some lines of the equal values k. Unstable regions are arranged/located about values of $a=n^2$. Since the cyclic modulation frequency of the parameter in equation (2.108) is equal to 2π, therefore, parametric excitation takes place when the average frequency of the oscillating system is close to half the integral values of the frequency of the parametric effect. This condition of parametric resonance is correct for the general case of the equation of Mathieu- Hill.

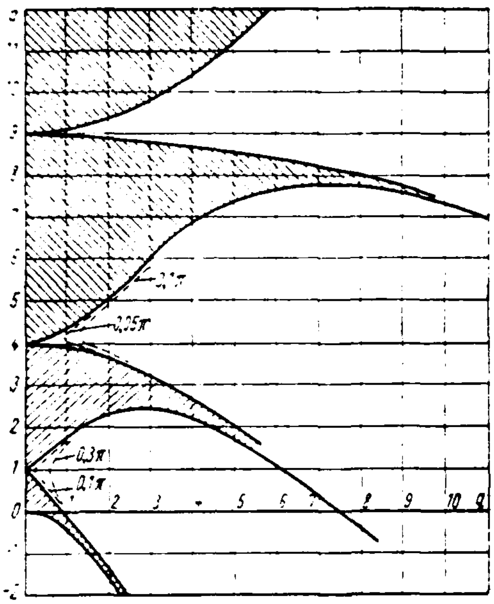

Fig. 2.11.

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§2.4. Beam of particles in the strong-focusing channel.

During the identification of the parameters of the strong-focusing channel must be first of all provided the stability of trajectories. For all further calculations of the focusing properties of channel are most interesting not individual particle trajectories, but envelope of particles, since precisely they determine the sizes/dimensions of the focused beam at each point of channel. It is obvious that the sizes/dimensions of beam in the channel must be connected with the acting focusing forces and with the value of the phase volume of beam. The behavior of beam in the channel depends on initial entrance conditions. Therefore direct problem of calculation consists in the determination envelope of particles from the initial conditions in the assigned focusing fields. Of no less significance is the inverse problem which is called the goal of the agreement: to determine initial conditions, with which the beam in the channel will have assigned optimum envelopes. Speaking about the initial conditions for the beam, we have in mind size/dimension and inclination/slope of envelope in each of the transverse planes or other values, with them unambiguously connected.

The phase volume of beam can be measured by different methods. One of the possible methods consists of the following: with the aid of the eyelet in the diaphragm they cut out the part of the beam, which corresponds to some values of transverse coordinates, and determine - on the disagreement of particles in drift space after the diaphragm - the scatter of path inclinations in the cut out part of the beam. If we consecutively/serially move the opening/aperture of diaphragm over entire beam section, then it is possible thus to find complete four-dimensional phase volume and distribution of phase density in it [61, 62].

Since into the equations of transverse vibrations is introduced the dimensionless independent variable τ, it is convenient to use phase plane with the coordinate axes x, $dx/d\tau$. The derivative $dx/d\tau$ is connected with the path inclination with the relationship/ratio

$$\frac{dx}{d\tau}=S\frac{dx}{dz}. \tag{2.109}$$

Let be known the projection of the transverse phase volume of beam on plane x, $dx/d\tau$ and this projection is limited by ellipse (Fig. 2.12)

$$ax^2+b\left(\frac{dx}{d\tau}\right)^2+2cx\frac{dx}{d\tau}=1. \tag{2.110}$$

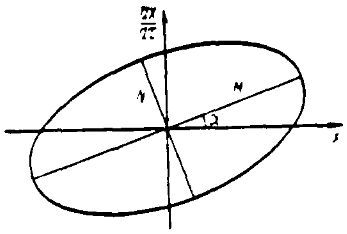

Fig. 2.12.

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Designations for the semi-axes and the angle of the slope of ellipse are clear from Fig. 2.12. The area, occupied with the projection of phase volume, let us designate πF_0, where

$$F_0 = MN \qquad (2.111)$$

there is a product of the semi-axes of ellipse (2.110). To avoid confusion let us name/call value F_0 the reduced volume of beam. If E - emittance of beam (2.3), then, according to equations (2.109), (2.111),

$$F_0 = SE. \qquad (2.112)$$

Combining equalities (2.4) and (2.112), we obtain also connection/communication between the transverse phase volume of beam (2.2) and reduced volume (2.111)

$$V_{\text{п}} = \frac{\gamma}{cT_{\Phi}} F_0. \qquad (2.113)$$

Since transverse phase volume - value invariant, then in the linear accelerators when T_Φ=const reduced volume in the nonrelativistic approximation/approach is also invariant. It is easy to show, converting equation (2.110) to the canonical form, that the reduced volume is connected with the coefficients of equation (2.110) with the equality

$$F_0 = \frac{1}{\sqrt{ab - c^2}}. \qquad (2.114)$$

Let us return now to the equation of individual trajectory, expressed through modulus/module and phase of the arbitrary pair of complex conjugate fundamental solutions (2.67),

$$x = A\sigma\cos(\psi+\Theta);$$
$$\frac{dx}{d\tau} = A\frac{d\sigma}{d\tau}\cos(\psi+\Theta) - \frac{A}{\sigma}\sin(\psi+\Theta). \qquad (2.115)$$

Expressions (2.115) are the equations of phase particle trajectory in the parametric form. Modulus/module and phase of fundamental solution are assumed the assigned functions of the parameter of equations τ. Values A and Θ depend only on initial conditions and are constants of motion. Eliminating from equations (2.115) trigonometric functions, we obtain expression for constant of motion A^2

$$A^2 = \left(\sigma\frac{dx}{d\tau} - \frac{d\sigma}{d\tau}x\right)^2 + \left(\frac{x}{\sigma}\right)^2. \qquad (2.116)$$

To each value/significance of constant of motion A^2 at current time τ corresponds on plane x, $dx/d\tau$ certain ellipse.

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Let us select initial conditions for the modulus/module $\sigma(0)$, $d\sigma/d\tau(0)$ and constant value A in such a way that ellipse (2.116) would coincide at zero time with the boundary of phase volume (2.110)

$$\begin{gathered}\frac{1}{A^2}\left[\left(\frac{d\sigma}{d\tau}\right)^2+\frac{1}{\sigma^2}\right]=a;\\ \frac{\sigma^2}{A^2}=b;\\ -\frac{1}{A^2}\sigma\frac{d\sigma}{d\tau}=c.\end{gathered} \tag{2.117}$$

Then to all particles, that lie on the boundary of phase volume, will correspond one and the same value/significance of constant of motion A and at any following moment of time ellipse (2.116) will coincide with the boundary of phase volume. After substituting expressions (2.117) into equality (2.114), we will obtain

$$A=\sqrt{F_0}. \tag{2.118}$$

Hence

$$\sigma(0)=\sqrt{bF_0};\quad \frac{d\sigma}{d\tau}(0)=-\frac{cF_0}{\sigma(0)}. \tag{2.119}$$

Expressions (2.119) can be represented also in other form, if we replace the coefficients of equation (2.110) with semi-axes and by the inclination/slope of the boundary ellipse (see Fig. 2.12):

$$\sigma(0) = \sqrt{\frac{M}{N}\cos^2\alpha + \frac{N}{M}\cdot\sin^2\alpha};$$
$$\frac{d\sigma}{d\tau}(0) = \frac{1}{2\sigma(0)}\left(\frac{M}{N} - \frac{N}{M}\right)\sin 2\alpha. \qquad (2.119a)$$

On envelope of particles lie/rest the particles, which have at the given moment/torque maximum misalignment. Since ellipse (2.116) with satisfaction of conditions (2.118), (2.119) always coincides with the boundary of phase volume, we are guaranteed, that to any value/significance of phase Θ corresponds the representative point of any particle of beam, which lies on the boundary. For the given moment of time always will be located the particle with phase Θ, with which $\cos(\psi - \Theta) = 1$. Hence, according to expressions (2.115), (2.118), it follows that envelope of particles takes the form

$$r(\tau) = \sqrt{F_0}\,\sigma(\tau). \qquad (2.120)$$

Thus, envelope of particles in each of the transverse planes is proportional to the modulus/module of certain complex solution of linear equations of motion. Proportionality factor with the standardized/normalized modulus/module - this square root of the reduced volume of beam. The corresponding modulus/module is the solution of equation (2.69) under initial conditions (2.119a), determined by form and location of the ellipse, which limits the projection of phase volume.

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Introducing expression (2.120) into formulas (2.69), (2.119a) we directly obtain equation and initial conditions for the envelope

$$\begin{gathered}\frac{d^2 r_x}{d\tau^2}+Q_x(\tau)\, r_x-\frac{F_{0x}^2}{r_x^3}=0;\\ r_x(0)=\sqrt{M_x^2\cos^2\alpha_x+N_x^2\sin^2\alpha_x};\\ \frac{dr_x}{d\tau}(0)=\frac{1}{2r_x(0)}(M_x^2-N_x^2)\sin 2\alpha_x.\end{gathered}\qquad (2.121)$$

For plane YOZ occur the same equations with the replacement of index.

If the phase volume of beam has finite quantity ($F_0\neq 0$), then equation for the envelope (2.121) does not coincide with the equation of trajectory (2.53). Consequently, in the beam with the final phase volume there is no particles whose trajectory would coincide with the envelope. "boundary particle" can exist only in the beams with the zero phase volume. As is known, hypothesis about the existence of boundary particle brings to so-called Brillouin flows [63]. Hence it is apparent that the Brillouin flow corresponds to particle motion in the beam where there is no disordered scatter of velocities [64-66]. Let us note that when deriving the equations (2.121) nowhere was utilized the assumption about the periodicity of functions $Q_x(\tau)$, $Q_y(\tau)$. This fact has the vital importance in the examination of beams with the noticeable space charge.

Let us assume that the phase volume of beam by the previous optic/optics is converted to the input in such a way that initial conditions (2.119a) would answer the periodic solution of equation for modulus/module (2.69). Then envelope of particles (2.120) is proportional to the modulus/module of Floquet's function

$$r(\tau)=\sqrt{F_0}\varrho(\tau). \qquad (2.122)$$

Envelope of particles in each of the transverse planes prove to be periodic functions with the period, equal to the period of focusing field. This beam is called matched with the channel. The ellipses, which limit the projections of the phase volume of matched beam, let us name/call Floquet's ellipses. The coefficients of equation (2.110), which is determining Floquet's ellipse, these are the periodic functions of the time

$$a=\frac{1}{F_0}\left[\left(\frac{d\varrho}{d\tau}\right)^2+\frac{1}{\varrho^2}\right];$$
$$b=\frac{1}{F_0}\varrho^2; \qquad (2.123)$$
$$c=-\frac{1}{F_0}\varrho\frac{d\varrho}{d\tau}.$$

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To each phase of the period of focusing field $0\leqslant\tau_0\leqslant 1$ corresponds its ellipse of Floquet, unambiguously connected with the parameters of the focusing channel $\rho(\tau_0)$, $dq/d\tau(\tau_0)$. Floquet's ellipse tests/experiences the identity transformation through the period of focusing field. The most characteristic phases of period - points at which the modulus/module of Floquet's function has an extremum. At these points first-order derivative of modulus/module is equal to zero, so that the coordinate axes of phase plane coincide with the axes of Floquet's ellipse. At the point of the maximum of Floquet's modulus/module the envelope has maximum value, and the instantaneous frequency of transverse vibrations (2.103) is minimum. Fig. 2.13 schematically presents the envelopes of matched beam in two mutually perpendicular planes, beam sections and ellipses of Floquet at some points of period. At points the II and IV moduli/modules of Floquet's functions have an extremum.

Let us substitute into equality (2.122) the instantaneous frequency of transverse vibrations (2.103)

$$r(\tau)=\sqrt{\frac{F_0}{T_\Phi\omega_r(\tau)}}. \qquad (2.124)$$

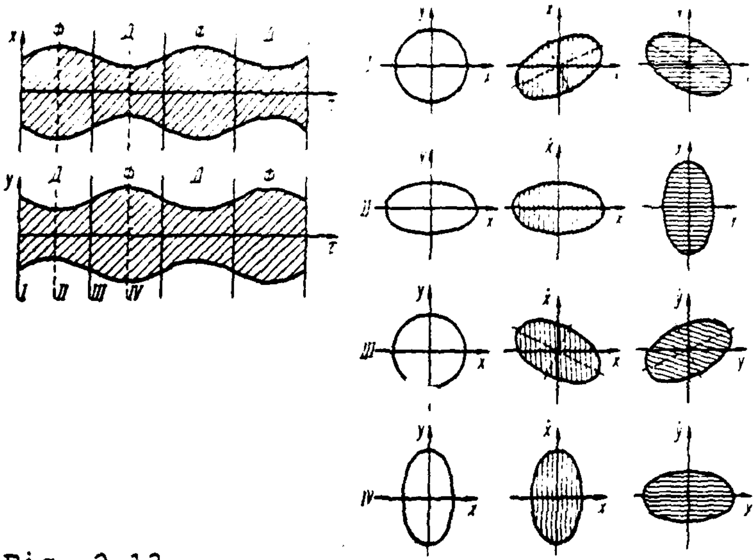

Fig. 2.13.

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The envelope of matched beam at each point of period is inversely proportional to square root from the instantaneous value/significance of frequency. The same relationship/ratio remains valid, if instantaneous frequency in the assigned phase of period adiabatically changes along the axis of channel. Replacing in expression (2.124) the reduced volume of beam with the value of transverse phase volume (2.113), we obtain the following connection/communication between the instantaneous frequency, the semi-axis of section and the phase volume of the beam

$$V_n' = \frac{1}{c}\gamma\omega_r r^2. \qquad (2.125)$$

Channel capacity let us name/call a maximally possible phase volume of the matched beam, still passed by channel. Generally, the fluctuation of particles in the channel - this is the sum of the free oscillations, which depend on initial conditions and determined for the ideal channel, and forced oscillations, caused by errors focusing in the system. Let a - a radius of that part of the aperture which is abstracted/removed under the free fluctuations of particles. The semi-axis of section reaches maximum value/significance $r = r_{макс}$ when $\omega_r = \omega_{r\ мин}$. Matched beam is passed without the losses through the channel, if $r_{макс} \leqslant a$. Hence we obtain formula for the channel capacity:

$$V_к = \frac{1}{c}\gamma\omega_{r\ мин} a^2. \qquad (2.126)$$

With the given aperture of channel and energy of particles the channel capacity is the higher, the greater the minimum instantaneous value/significance of the frequency of transverse vibrations in the period of focusing field.

Channel capacity they frequently prefer to characterize with the value of acceptance, equal maximum to the permissible emittance of the matched beam, still passed by channel. The acceptance of channel is connected with the capacity with the relationship/ratio, analogous to expression (2.4):

$$A = \frac{V_к}{\beta\gamma}. \qquad (2.127)$$

After using expression (2.103), let us represent capacity and acceptance of channel in the form

$$V_к = \frac{\gamma}{cT_\Phi \varrho^2_{макс}} a^2;$$
$$A = \frac{1}{vT_\Phi \varrho^2_{макс}} a^2. \qquad (2.128)$$

Hence it follows that in the linear accelerators when $T_\Phi = \text{const}$ the channel capacity in the nonrelativistic approximation/approach is invariant, and in the relativistic approximation/approach it slowly increases proportional to Lorenz's factor. The acceptance of channel in the linear accelerators when $T_\Phi = \text{const}$ depends substantially on the energy of particles and is inconvenient as the characteristics of the focusing system of accelerator.

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In the circular accelerators is always kept constant the length of the period of focusing field $S = vT_\Phi$, so that acceptance does not depend on energy of particles. Therefore in the theory of the circular accelerators they prefer to use the concept of the

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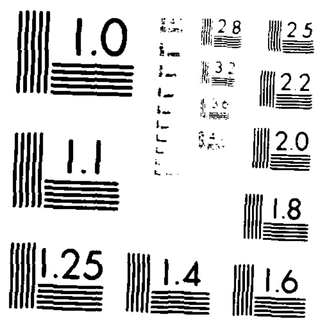

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acceptance of channel and respectively by the emittance of beam.

Let us assume the transverse phase volume of beam at the entrance of the strong-focusing channel is limited by the ellipse, which does not coincide with Floquet's ellipse the beginning of channel. If they are known to Floquet's function for this channel, then the envelope of unmatched beam can be found without the integration of equation (2.69). Let us examine certain complex solution of equation of motion whose modulus/module the unknown function $\sigma(\tau)$,

$$\chi(\tau)=\sigma e^{i\psi}=c_1\varphi(\tau)+c_2\varphi^*(\tau), \qquad (2.129)$$

where c_1, c_2 - complex constants:

$$c_1=C_1e^{i\gamma_1}; \quad c_2=C_2e^{i\gamma_2}. \qquad (2.130)$$

Since function $\chi(\tau)$ is determined with an accuracy to constant phase, we can assume $\gamma_1=\gamma$; $\gamma_2=0$. Moduli/modules C_1 and C_2 are connected with the the identical standardization of all complex conjugate solutions of equation of motion. Since

$$\begin{vmatrix} \chi & \chi^* \\ \frac{d\chi}{d\tau} & \frac{d\chi^*}{d\tau} \end{vmatrix}=(c_1c_1^*-c_2c_2^*)\begin{vmatrix} \varphi & \varphi^* \\ \frac{d\varphi}{d\tau} & \frac{d\varphi^*}{d\tau} \end{vmatrix},$$

then

$$C_1^2-C_2^2=1. \qquad (2.131)$$

Further, substituting formulas (2.129), (2.130) in the expression for the modulus/module

$$\sigma^2 = \chi\chi^{*}.$$

we obtain

$$\sigma(\tau) = \varrho(\tau)\sqrt{C_1^2 + C_2^2 - 2C_1C_2\cos[2\Phi(\tau) - \gamma]}. \qquad (2.132)$$

Thus, the envelope of unmatched beam is the product of periodic processes with two frequencies: with the repetition frequency of the focusing structure and with a double frequency of transverse vibrations of $2\omega_r$ or on the average during the period of focusing field $\frac{2\mu}{T_\Phi}$. If the relation of these frequencies is great, then the envelope of unmatched beam has maximums in each period of structure (Fig. 2.14). These maximums subsequently let us name/call the local maximums of envelope. But if both frequencies are sufficiently close, then in some periods of focusing field local maximums can be absent.

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At those points of the channel where

$$\cos(2\Phi + \gamma) = \pm 1, \qquad (2.133)$$

we have

$$\sigma(\tau)=(C_1\pm C_2)\,\varrho(\tau).$$

Since $C_1>0$; $C_2>0$, then $C_1-C_2<C_1+C_2$. From condition (2.131) it follows that

$$C_1+C_2>1;$$
$$0<C_1-C_2<1.$$

Virtually both frequencies of envelope never are located in rational sense; therefore in the sufficiently long channel to point (2.133) it is necessary the maximum of Floquet's modulus/module. At this point the envelope of unmatched beam will be greatest

$$\sigma_{\text{макс}}=(C_1+C_2)\,\varrho_{\text{макс}}. \qquad (2.134)$$

The maximum of envelope (2.134) let us name/call principal maximum. The local maximums of envelope oscillate, having approximately a period, which corresponds to the double frequency of transverse vibrations $2\mu/T_\Phi$. The greatest values of local maximums in each period of their oscillations are approximately equal to principal maximum. Therefore latter/last maximum can be the adequate/approaching practical criterion for evaluating the sizes/dimensions of unmatched beam in the strong-focusing channel. From equality (2.134) it follows that the principal maximum of the envelope of unmatched beam always exceeds the maximum of the envelope of matched beam with the same value of transverse phase volume.

The obtained result has simple geometric interpretation. Since discrete/digital points in the trajectory, distant for the period of focusing field, lie/rest on sinusoid (2.107), it is obvious that the phase trajectories of the representative points, examined/considered through the period coincide with Floquet's ellipses. Let us describe Floquet's ellipse around the phase volume of unmatched beam (Fig. 2.15). For the certainty let us examine the discrete/digital points, which correspond the maximum of Floquet's modulus/module.

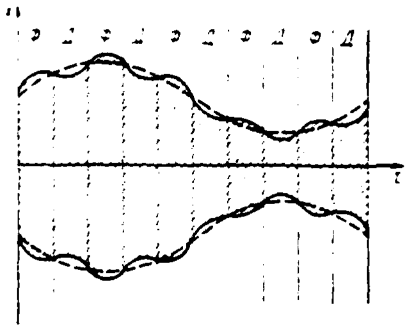

Fig. 2.14.

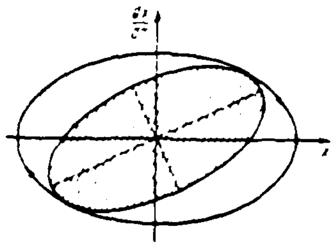

Fig. 2.15.

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The phase volume of unmatched beam will rotate within the described ellipse of Floquet, so that the semi-axis of section will fluctuate with the double frequency of rotation. The maximum size of unmatched beam proves to be by the same as the size/dimension of matched beam with the phase volume, which corresponds to the area of the described ellipse of Floquet. The area of the described ellipse of Floquet can be named/called the effective phase volume of beam.

Let us determine the principal maximum of the envelope of unmatched beam under the assigned initial conditions $\sigma(0)$, $d\sigma/d\tau$ (0) (2.119a) and with the assigned functions $\rho(\tau)$, $d\rho/d\tau(\tau)$. Differentiating the square of expression (2.132) and assuming/setting $\tau=0$, taking into account expressions (2.131), (2.132) we obtain three equations for definition three the unknowns C_1, C_2, γ:

$$\begin{aligned} 2C_1C_2\sin\gamma &= \frac{\sigma}{\varrho}\left(\sigma\frac{d\varrho}{d\tau}-\varrho\frac{d\sigma}{d\tau}\right)_{\tau=0}; \\ C_1^2+C_2^2-2C_1C_2\cos\gamma &= \left(\frac{\sigma}{\varrho}\right)^2_{\tau=0}; \qquad (2.135) \\ C_1^2-C_2^2 &= 1. \end{aligned}$$

When

$$(C_1+C_2)^2=\xi. \qquad (2.136)$$

Then system of equations (2.135) is reduced to the form

$$\begin{aligned} \left(\xi-\frac{1}{\xi}\right)\sin\gamma &= 2\frac{\sigma}{\varrho}\left(\sigma\frac{d\varrho}{d\tau}-\varrho\frac{d\sigma}{d\tau}\right)_{\tau=0}; \qquad (2.137) \\ \xi-\left(\xi-\frac{1}{\xi}\right)\sin^2\frac{\gamma}{2} &= \left(\frac{\sigma}{\varrho}\right)^2_{\tau=0}. \end{aligned}$$

Eliminating γ from equations (2.137), we obtain

$$\xi^2-C\xi+1=0. \qquad (2.138)$$

where

$$C=\left(\frac{\sigma}{\varrho}\right)^2+\left(\frac{\varrho}{\sigma}\right)^2+\left(\sigma\frac{d\varrho}{d\tau}-\varrho\frac{d\sigma}{d\tau}\right)^2_{\tau=0} \qquad (2.139)$$

If beam is matched with the channel, then C=2 and $\xi_1=\xi_2=1$. Under any other initial conditions $\xi_1>1$; $\xi_2<1$. Principal maximum of the unmatched beam

$$r_{\text{макс}}=\sqrt{\xi_1F_0}\,\varrho_{\text{макс}}. \qquad (2.140)$$

The value of root $\xi_1>1$ or unambiguously with it the connected value of parameter C>2 can serve as the criterion of agreement.

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Let us assume that the matched beam must have at the entrance of channel a crossover: $d\rho/d\tau(0)=0$. Further let us assume that the unmatched beam has at the entrance parallel envelope, but the semi-axes of section differ from those matched. Then

$$C=\left(\frac{\sigma}{\varrho}\right)^2_{\tau=0}+\left(\frac{\varrho}{\sigma}\right)^2_{\tau=0}.$$

From equation (2.138) follows

$$\xi_1=\begin{cases}\frac{\sigma(0)}{\varrho(0)} & \text{при}^{(1)}\ \sigma(0)>\varrho(0);\\ \frac{\varrho(0)}{\sigma(0)} & \text{при}^{(1)}\ \sigma(0)<\varrho(0).\end{cases}$$

Key: (1). with.

Thus, for instance, maximum of unmatched beam exceeds the maximum of matched beam two times, if $\sigma(0)=2\rho(0)$ or $\sigma(0)=1/2\rho(0)$. Agreement to always more easily rate/estimate, if a beam has at the entrance of channel crossover. Otherwise the goal substantially is complicated, since besides the section it is necessary to measure the angles of the slope of envelope.

Let the entrance of the strong-focusing channel enter the unmatched beam, which reaches in the channel of radius a, equal to the maximum permissible value/significance of the amplitude of free oscillations. Then, according to expressions (2.122), (2.140), the maximum size of matched beam with the same transverse phase volume is equal to

$$r_{\text{макс (согл)}} = \frac{a}{\sqrt{\xi_1}}.$$

It is hence easy to find the phase volume of the unmatched beam

$$V_{\text{н}} = \frac{1}{c}\,\gamma\omega_{r\,\text{мин}} a^2 \frac{1}{\xi_1}.$$

Thus, the greatest transverse phase volume of the unmatched beam which can be passed through the focusing channel, is connected with the channel capacity as follows

$$V_{\text{н макс}} = \frac{V_{\text{к}}}{\xi_1}. \qquad (2.141)$$

Consequently, the best beam focusing in the channel is reached when beam is preliminarily prepared by the system of matching lenses. For determining the matched initial conditions it is necessary to know the phase volume of beam and function of Floquet of the channel

$$r(0) = \sqrt{F_0}\,\varrho(0); \quad \frac{dr}{d\tau}(0) = \sqrt{F_0}\,\frac{d\varrho}{d\tau}(0). \qquad (2.142)$$

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Since values F_{0x}, F_{0y} during the conversions of phase volume remain constants, the goal of beam matching with the channel is reduced to the transformation of four parameters of beam $\sigma_x, \sigma_y, \frac{d\sigma_x}{d\tau}, \frac{d\sigma_y}{d\tau}$ or that the same, $M_x, M_y, \alpha_x, \alpha_y$. Therefore a minimum number of independent parameters of the matching chain/network of lenses in the general case must be equal to four.

§ 2.5. Matrix method of calculation of Floquet's functions.

Above was explained the important role of Floquet's modulus/module as the function, which describes the envelope of matched beam in focusing channel [78]. The values of the modulus/module of Floquet and his derivative at the entrance of channel initially assign for the matched beam, which in the principle gives the possibility to calculate the matching optic/optics before the channel. Floquet's modulus/module uniquely determines the instantaneous frequency of transverse vibrations of particles and, therefore, channel capacity with the assigned aperture. This makes it possible to formulate the requirements, presented to the phase volume of beam. But if the phase volume of beam is assigned, then Floquet's modulus/module determines the maximum size of the matched or unmatched beam in this channel. Of greatest interest are the values of the modulus/module of Floquet and his derivative at the entrance of channel, and also the value of Floquet's modulus/module at the point of maximum.

For calculating Floquet modulus and its derivative at any assigned point of period it is convenient to utilize an apparatus of matrix algebra [67]. let us divide the period of focusing field in the sections, for each of which the particle motion is described by its linear equations (accelerating gap, quadrupole lens, idle gap/interval, etc.). Solution of equations of motion in each section, determined under the arbitrary initial conditions, gives the matrix/die of linear transformation for displacement and path inclination from the beginning toward the end of the section. Multiplying the matrices/dies of individual sections, we obtain the complete matrix/die of period whose elements/cells are connected with the values of Floquet functions and do not depend on the selection of particular solutions. This makes it possible to avoid the basic difficulty of calculating Floquet's modulus/module from nonlinear equation (2.69), which consists in finding of initial conditions for the periodic solution. The task of determining the matrices/dies is simplified, if the fields of quadrupole lenses are approximated by "square wave".

Let us examine the arbitrary real solution of equation of motion, expressed through Floquet's functions [68]:

$$x(\tau)=b_1\varphi_1(\tau)+b_2\varphi_2(\tau).$$

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In the stable region the functions of Floquet and respectively arbitrary constants are complex conjugated, while in the unstable region - actual. Connection/communication between the values of functions x, $dx/d\tau$ at the moment of time τ and the values of arbitrary constants can be presented in the matrix form

$$\begin{pmatrix} x \\ \frac{dx}{d\tau} \end{pmatrix}_\tau = \begin{pmatrix} \varphi_1 & \varphi_2 \\ \frac{d\varphi_1}{d\tau} & \frac{d\varphi_2}{d\tau} \end{pmatrix}_\tau \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \qquad (2.143)$$

Let us solve matrix equation (2.143) relative to the column of the arbitrary constants

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 & \varphi_2 \\ \frac{d\varphi_1}{d\tau} & \frac{d\varphi_2}{d\tau} \end{pmatrix}_\tau^{-1} \begin{pmatrix} x \\ \frac{dx}{d\tau} \end{pmatrix}_\tau. \qquad (2.144)$$

If we now write equation (2.143) for the moment/torque of time $\tau+1$ and to replace the column of arbitrary constants with the right side of equality (2.144), then we will obtain

$$\begin{pmatrix} x \\ \frac{dx}{d\tau} \end{pmatrix}_{\tau+1} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}_\tau \begin{pmatrix} x \\ \frac{dx}{d\tau} \end{pmatrix}_\tau, \qquad (2.145)$$

where

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} \varphi_1 & \varphi_2 \\ \frac{d\varphi_1}{d\tau} & \frac{d\varphi_2}{d\tau} \end{pmatrix}_{\tau+1} \begin{pmatrix} \varphi_1 & \varphi_2 \\ \frac{d\varphi_1}{d\tau} & \frac{d\varphi_2}{d\tau} \end{pmatrix}_\tau^{-1} \qquad (2.146)$$

Let us assume $\tau=n+\tau'$, where n - integer, and τ' - any phase of period, which lies within limits of $0\leqslant\tau'\leqslant 1$. Since the initial equation of Mathieu-Hill is invariant relative to transformation $\tau=n+\tau'$, then with any n can be selected one and the same pair of characteristic functions. Consequently, the matrix/die

$$T=\begin{pmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{pmatrix} \qquad (2.147)$$

does not depend on n. Matrix elements (2.147) - the function of the selected phase of period τ. Matrix/die (2.147) is called the matrix/die of the period of the focusing field, which begins from the chosen phase τ'. Determinant of matrix/die in the right side of equation (2.143) - the Wronskian determinant of Floquet's functions

$$\begin{vmatrix}\varphi_1 & \varphi_2\\ \dot{\varphi}_1 & \dot{\varphi}_2\end{vmatrix}_{\tau}=W(\tau).$$

For the reciprocal matrix occurs the relationship/ratio

$$\left|\begin{pmatrix}\varphi_1 & \varphi_2\\ \dot{\varphi}_1 & \dot{\varphi}_2\end{pmatrix}_{\tau}^{-1}\right|=\frac{1}{W(\tau)}.$$

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According to expression (2.146),

$$T = \frac{W(\tau - 1)}{W(\tau)}.$$

According to the theorem about the constancy of Wronskian determinant $T = 1$:

$$T_{11}T_{22} - T_{12}T_{21} = 1. \qquad (2.148)$$

Let us expand/develop matrix product (2.146). Utilizing the Floquet theorem (2.88), (2.89), we obtain

$$\left.\begin{aligned} T_{11} &= \frac{1}{W}\left(e^{l}\varphi_1\frac{d\varphi_2}{d\tau} - e^{-l}\varphi_2\frac{d\varphi_1}{d\tau}\right); \\ T_{12} &= -\frac{2}{W}\varphi_1\varphi_2\,\mathrm{sh}\,l; \\ T_{21} &= \frac{2}{W}\cdot\frac{d\varphi_1}{d\tau}\cdot\frac{d\varphi_2}{d\tau}\,\mathrm{sh}\,l; \\ T_{22} &= \frac{1}{W}\left(e^{-l}\varphi_1\frac{d\varphi_2}{dt} - e^{l}\frac{d\varphi_1}{d\tau}\varphi_2\right). \end{aligned}\right\} \qquad (2.149)$$

Matrix elements of period - always real numbers. In fact, in the unstable region, according to expression (2.90),

$$W = 1;\ e^{l} = (-1)^{n}e^{k};\ \mathrm{sh}\,l = (-1)^{n}\,\mathrm{sh}\,k,$$

so that all factors in expressions (2.149) are real. In the stable region

$$W = -2i;\ \mathrm{sh}\,l = i\sin\mu,$$

whence it is directly evident that T_{12}, T_{21} are real. Formulas for elements/cells T_{11}, T_{22} contain in the parenthesis of a difference in the complex conjugate numbers; therefore T_{11}, T_{22} is also real.

From equalities (2.149) it follows that the sum of the elements/cells of principal diagonal is equal to

$$T_{11}+T_{22}=2\operatorname{ch} l. \qquad (2.150)$$

Hence

$$\operatorname{ch} l=\frac{1}{2}\,SpT. \qquad (2.150a)$$

From inequality (2.91) it follows that the stability of trajectories in the strong-focusing channel is provided with satisfaction of the condition

$$SpT<2. \qquad (2.151)$$

In the region of stable solutions $l=i\mu$ and equality (2.150) has the form

$$\cos\mu=\frac{T_{11}+T_{22}}{2}, \qquad (2.152)$$

where μ - phase change of Floquet in one period of focusing field.

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In view of the fact that the matrix elements of period (2.147) are connected with condition (2.148), only three elements/cells can be determined independently. It is expedient to introduce three independent variables, which are determining the matrix/die of period, as follows:

$$\cos\mu=\frac{T_{11}+T_{22}}{2};\quad \nu=\sqrt{-\frac{T_{21}}{T_{12}}};\quad \sin\varepsilon=\frac{T_{11}-T_{22}}{2\sqrt{-T_{12}T_{21}}}. \qquad (2.153)$$

In the region of stable solutions parameters ν, ε are real. Specifically,, conditions (2.148), (2.151) lead to the inequality

$$(T_{11}+T_{22})^2=(T_{11}-T_{22})^2+4+4T_{12}T_{21}<4,$$

or

$$-T_{12}T_{21}>\frac{1}{4}(T_{11}-T_{22})^2.$$

Thus, in the stable region the elements/cells of nonmajor diagonal always have equal signs, and $\sin\varepsilon$ on the modulus/module is less than unity.

From equations (2.153) can be obtained the expressions for the matrix elements of the period through independent parameters of the matrix/die

$$T_{11}=\frac{\cos(\mu-\varepsilon)}{\cos\varepsilon};\quad T_{12}=\frac{\sin\mu}{\nu\cos\varepsilon};$$
$$T_{21}=-\frac{\nu\sin\mu}{\cos\varepsilon};\quad T_{22}=\frac{\cos(\mu+\varepsilon)}{\cos\varepsilon}. \tag{2.154}$$

Let us connect the parameters of matrix/die ν, ε with the modulus/module of Floquet function of channel. Substituting expressions (2.149) in the right sides of formulas (2.153) and taking into account designation for Floquet's functions in stable region (2.96), we obtain

$$\nu=\frac{1}{\varrho}\left[\left(\frac{d\varrho}{d\tau}\right)^2+\frac{1}{\varrho^2}\right]^{1/2};$$
$$\sin\varepsilon=-\frac{d\varrho}{d\tau}\left[\left(\frac{d\varrho}{d\tau}\right)^2+\frac{1}{\varrho^2}\right]^{-1/2}. \tag{2.155}$$

Of three independent parameters of matrix/die T one parameter - constant value: μ=const, and two others $\nu(\tau)$, $\varepsilon(\tau)$ — periodic functions of the independent variable with the period $\Delta\tau$=1. Parameter ε is equal to zero at the points of the extremum of Floquet's modulus/module. At the points of extremum the matrix/die of period is simplified:

$$T=\begin{pmatrix}\cos\mu & \frac{1}{\nu}\sin\mu\\ -\nu\sin\mu & \cos\mu\end{pmatrix}. \tag{2.154a}$$

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According to expression (2.155),

$$\operatorname{tg} \varepsilon = -\varrho \frac{d\varrho}{d\tau};$$
$$\nu \cos \varepsilon = \frac{1}{\varrho^2}. \qquad (2.156)$$

Hence, taking into account expression (2.100), we have

$$\frac{d\Phi}{d\tau} = \nu \cos \varepsilon. \qquad (2.157)$$

Product $\nu \cos \varepsilon$ — the instantaneous frequency of transverse oscillations to scale of time τ. To scale of time t

$$\omega_r = \frac{\nu \cos \varepsilon}{T_\Phi}. \qquad (2.158)$$

Channel capacity is determined by the minimum value/significance of instantaneous frequency (2.158). At the point where frequency (2.158) reaches minimum value/significance, $\varepsilon = 0$ and

$$\omega_{r\,мин} = \frac{\nu_\Phi}{T_\Phi}. \qquad (2.159)$$

Here ν_Φ — value of parameter ν in the middle of the focusing gap/interval with $\varepsilon = 0 (T_{11} = T_{22})$. Subsequently for the convenience we utilize dimensional (in scale of time t) and dimensionless (to scale of time τ) frequencies.

If the matrix/die of period is calculated, then means known the modulus/module of Floquet and his derivative at the appropriate point of the channel:

$$\varrho(\tau)=\frac{1}{\sqrt{v\cos e}};\qquad (2.160)$$
$$\frac{d\varrho}{d\tau}(\tau)=-\frac{\operatorname{tg} e}{\varrho(\tau)}.$$

Let us examine based on examples the procedure of calculation of the matrices/dies of period. in the channel of the type FOD (see Fig. 2.6) the accelerating clearance is located in the gap/interval, situated after the focusing lens, and between the pair of the lenses, structurally/constructurally united, is a section, free from the fields.

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At the length of the focusing lens the displacement and path inclination undergo linear transformation

$$\begin{pmatrix} x \\ \dot{x} \end{pmatrix}_1 = P\begin{pmatrix} x \\ \dot{x} \end{pmatrix}_0,$$

where x_0, $\dot{x}_0=dx/d\tau(0)$ - displacement and path inclination at the entrance, and x_1, $\dot{x}_1$ - at the output of lens; P - matrix/die, which characterizes the action of the focusing lens. After the accelerating clearance

$$\begin{pmatrix} x \\ \dot{x} \end{pmatrix}_2 = \Gamma\begin{pmatrix} x \\ \dot{x} \end{pmatrix}_1 = \Gamma P\begin{pmatrix} x \\ \dot{x} \end{pmatrix}_0,$$

where Γ – matrix/die of accelerating gap. Further particle passes the defocusing lens at output of which we have

$$\begin{pmatrix} x \\ \dot{x} \end{pmatrix}_3 = \bar{P}\begin{pmatrix} x \\ \dot{x} \end{pmatrix}_2 = P\Gamma P\begin{pmatrix} x \\ \dot{x} \end{pmatrix}_0,$$

where $\bar{P}$ - matrix/die of the defocusing lens. The period of focusing field concludes according to the condition with idle gap/interval; at the output of idle gap/interval the displacement and path inclination have values x_4, $\dot{x}_4 = dx/d\tau$(1), moreover

$$\begin{pmatrix} x \\ \dot{x} \end{pmatrix}_4 = H\begin{pmatrix} x \\ \dot{x} \end{pmatrix}_3 = H\bar{P}\Gamma P\begin{pmatrix} x \\ \dot{x} \end{pmatrix}_0.$$

Thus, matrix/die the period of focusing field takes the form

$$T = H\bar{P}\Gamma P. \qquad (2.161)$$

According to expression (2.161), the matrices/dies of sections are multiplied in backward sequence - from the end/lead of the period at the beginning. The matrix/die of the same period in the perpendicular plane will be, obviously, that follows

$$\bar{T} = HP\Gamma\bar{P}. \qquad (2.161a)$$

Now let us assume that we deal concerning the channel of the type FOF-DOD (see Fig. 2.6). Let the accelerating gaps be arranged/located between the lenses of one sign, and idle - between the lenses of different signs. Let us begin period from the middle of accelerating gap. Then

$$T = \Gamma_{1/2}PH\bar{P}\Gamma\bar{P}HP\Gamma_{1/2}. \qquad (2.162)$$

Index 1/2 means that is taken the matrix/die of half of accelerating gap. Analogously it is possible to compose matrices/dies for any other types of period. If the matrix product of individual sections is symmetrical as, for example, in expression (2.162), then the elements/cells of the principal diagonal of matrix/die T are equal, so that at the initial point of period occurs the extremum of Floquet's modulus/module.

Let us determine the matrices/dies of sections.

Matrix/die of quadrupole lens. Let us assume that at the length of the section, occupied with quadrupole lens, there is no accelerating field. Then, according to expressions (2.55), (2.56),

$$Q=S^2\frac{eG}{p}. \tag{2.163}$$

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Let us assume further that at the length of lens D the gradient of focusing field - constant value. In this case of Q=const the equation of motion (2.53) has in the focusing section a solution

$$x=A\cos\sqrt{Q}\tau-B\sin\sqrt{Q}\tau;$$
$$\frac{dx}{d\tau}=-A\sqrt{Q}\sin\sqrt{Q}\tau+B\sqrt{Q}\cos\sqrt{Q}\tau.$$

where A, B - arbitrary constants. Let us assume at the entrance of the focusing lens correspond $\tau=0$; the coordinate of output $\tau=D/S$. Arbitrary constants are determined by initial entrance conditions

$$A = x_0;\ B = \frac{1}{\sqrt{Q}} \left(\frac{dx}{d\tau} \right)_0 .$$

At the output of lens we have

$$\begin{aligned} x_1 &= x_0 \cos \frac{D\sqrt{Q}}{S} + \frac{\dot{x}_0}{\sqrt{Q}} \sin \frac{D\sqrt{Q}}{S}; \\ \dot{x}_1 &= -\dot{x}_0 \sqrt{Q} \sin \frac{D\sqrt{Q}}{S} + \dot{x}_0 \cos \frac{D\sqrt{Q}}{S}. \end{aligned} \quad (2.164)$$

Value

$$K = \frac{D\sqrt{Q}}{S}$$

let us name/call the hardness of lens. Substituting for Q its value/significance (2.163), we obtain the following expression for the hardness of the lens

$$K = D\sqrt{\frac{eG}{p}}. \quad (2.165)$$

In formula (2.165) all values to the right are measured in the system of SI units. Hardness is dimensionless value. According to expressions (2.164), (2.165), the matrix/die of the focusing lens takes the form

$$P = \begin{pmatrix} \cos K & \frac{D}{SK} \sin K \\ -\frac{SK}{D} \sin K & \cos K \end{pmatrix}. \quad (2.166)$$

In formula (2.165) the value of gradient is taken in terms of the absolute value. Then hardness proves to be one and the same real value in plane XOZ and in plane YOZ. For the defocusing lens the gradient has negative sign, and value K in matrix/die (2.166) should be replaced by iK.

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Since cos iK=ch K, sin iK=i sh K, then matric of the defocusing lens is equal to

$$\bar{P}=\begin{pmatrix} \operatorname{ch} K & \frac{D}{SK}\operatorname{sh} K \\ \frac{SK}{D}\operatorname{sh} K & \operatorname{ch} K \end{pmatrix}. \qquad (2.167)$$

Matrix/die of idle gap/interval. At the length of the idle gap/interval h of field are absent: $Q(\tau)\equiv 0$ and K=0. Let us pass in matrix (2.166) to the limit with $K\to 0$. Assuming/setting D=h, we obtain

$$H=\begin{pmatrix} 1 & h/S \\ 0 & 1 \end{pmatrix}. \qquad (2.168)$$

Matrix/die of accelerating gap in the system with the traveling wave. Let us examine the case when between the lenses particles are accelerated in the traveling wave. This case can be encountered in the linear accelerators to the superhigh energies. Within the accelerating gap according to the condition there are no focusing fields; therefore, according to expressions (2.55), (2.57),

$$Q(\tau)=\frac{eS^2}{2\mathscr{E}_0\beta^2\gamma^3}\cdot\frac{\partial E_z}{\partial z}(\tau).$$

Let us replace derivative of field with expression (2.12). Taking into account formulas (1.70), (1.82) we have

$$Q(\tau)=-2\pi^2\left(\frac{S}{\beta\lambda}\right)^2\left(\frac{\Omega}{\omega}\right)^2\frac{\sin\varphi}{\sin\varphi_s}.$$

Further let us assume that at the length of the accelerating gap between the adjacent lenses it is possible to disregard a change in the phase of particle φ. Then Q=const. Let us introduce designations for the following combinations of the parameters which will be frequently encountered subsequently:

$$\gamma_s=\pi^2\left(\frac{S}{\beta\lambda}\right)^2\left(\frac{\Omega}{\omega}\right)^2; \qquad (2.169)$$

$$\gamma=\gamma_s\frac{\sin\varphi}{\sin\varphi_s}. \qquad (2.170)$$

Value γ let us name/call the factor of defocusing. The equation of transverse motion for the particle with the phase φ in this accelerating gap will take the form

$$\frac{d^2x}{d\tau^2}-2\gamma x=0. \qquad (2.171)$$

Equation (2.171) is reduced to equation (2.13), if we assume $\varphi=\varphi_s$ and to switch over from variable/alternating τ to the dimensional time t.

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The matrix/die of accelerating gap at the positive value/significance of the factor of defocusing takes the form

$$\Gamma = \begin{pmatrix} \operatorname{ch} \frac{l\sqrt{2\gamma}}{S} & \frac{1}{\sqrt{2\gamma}} \operatorname{sh} \frac{l\sqrt{2\gamma}}{S} \\ \sqrt{2\gamma} \operatorname{sh} \frac{l\sqrt{2\gamma}}{S} & \operatorname{ch} \frac{l\sqrt{2\gamma}}{S} \end{pmatrix}. \qquad (2.172)$$

where l - length of accelerating gap. If particle moves in such phase of travelling waves, in which the factor of defocusing is negative, then accelerating field focuses; hyperbolic functions in matrix/die (2.172) convert/transfer with this kV circular ones. The determinant of matrix/die (2.172) just as the determinants of matrices/dies (2.166)-(2.168), is equal to unity. It is possible to show that the equality to one of all determinants of matrix of transformation - a consequence of Liouville's theorem.

For the high energy particles the frequency of longitudinal vibrations is small, so that the factor of defocusing (2.169) is substantially less than one. In this case matrix/die (2.172) is possible with accuracy sufficient for the practical purposes to write in the form

$$\Gamma = \begin{pmatrix} 1 & \frac{l}{S} \\ 2\gamma \frac{l}{S} & 1 \end{pmatrix}. \qquad (2.173)$$

Any matrix/die, whose elements/cells of principal diagonal are identical, and determinant is equal to one, it can be represented in the form of the product of following three simple matrices:

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{11} \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}. \qquad (2.174)$$

Producing multiplication and equalizing equivalent components of matrices/dies, we obtain

$$a = \frac{T_{11}-1}{T_{21}}; \quad b = T_{21}. \qquad (2.175)$$

Let us decompose on the factors of type (2.174) matrix/die (2.172):

$$a = \frac{1}{\sqrt{2\gamma}} \operatorname{th} \frac{l\sqrt{2\gamma}}{2S}; \quad b = \sqrt{2\gamma} \operatorname{sh} \frac{l\sqrt{2\gamma}}{S}. \qquad (2.176)$$

With $\gamma \ll 1$ we have $a = l/2S$, $b = 2\gamma(l/S)$ or

$$\Gamma = \begin{pmatrix} 1 & l/2S \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2\gamma\frac{l}{S} & 1 \end{pmatrix} \begin{pmatrix} 1 & l/2S \\ 0 & 1 \end{pmatrix}.$$

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Thus, if $\gamma \ll 1$, then the action of accelerating gap is reduced to the action of two idle clearances by length $l/2$ and to the action of the middle refracting plane with the refractive index of the trajectory

$$\frac{\Delta \dot{x}}{x} = 2\gamma \frac{l}{S}. \qquad (2.177)$$

Ratio S/l - number of accelerating gaps, which fall for one period of focusing field. Let us exclude the focusing lenses. Then the general/common/total effect of defocusing, connected with accelerating field, is characterized in the period only by the sequential action of matrices (2.172)

$$\Gamma_s = \begin{pmatrix} \operatorname{ch} \frac{l\sqrt{2\gamma}}{S} & \frac{1}{\sqrt{2\gamma}} \operatorname{sh} \frac{l\sqrt{2\gamma}}{S} \\ \sqrt{2\gamma} \operatorname{sh} \frac{l\sqrt{2\gamma}}{S} & \operatorname{ch} \frac{l\sqrt{2\gamma}}{S} \end{pmatrix}^{\frac{S}{l}}.$$

In accordance with the theorem of Sylvester [68]

$$\Gamma_s = \begin{pmatrix} \operatorname{ch} \sqrt{2\gamma} & \frac{1}{\sqrt{2\gamma}} \operatorname{sh} \sqrt{2\gamma} \\ \sqrt{2\gamma} \operatorname{sh} \sqrt{2\gamma} & \operatorname{ch} \sqrt{2\gamma} \end{pmatrix}. \qquad (2.178)$$

If the factor of defocusing is sufficiently small, then

$$\Gamma_s = \begin{pmatrix} 1 & 1 \\ 2\gamma & 1 \end{pmatrix}.$$

Hence we see that the refractive index of trajectory at the length of one period of focusing field is numerically equal to doubled value of the factor of defocusing.

Matrix/die of accelerating gap in the system with the standing waves. Let us examine for the concreteness of that accelerating the clearance between drift tubes. The calculation, given subsequently, can be transferred to any accelerating system with the standing waves. We approximate the accelerating field in the clearance by "square wave": let us assume that the longitudinal component of field at the gap length is constant, and within drift tube abruptly it drops up to zero. In view of the uniformity of field in the clearance function (2.57) is equal to zero out of the clearance and inside it, and entire effect is created only at the end-points at the entrance and at the output of clearance. The matrix/die of the accelerating clearance can be represented in the form of the product of three matrices/dies. Outer matrices/dies describe refraction of trajectory at the entrance and at the output of clearance, and average corresponds to the idle gap/interval with a length of g:

$$\Gamma = \begin{pmatrix} 1 & 0 \\ \gamma_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & g/S \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma_1 & 0 \end{pmatrix}.$$

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Hence

$$\Gamma = \begin{pmatrix} 1 - \frac{g}{S}\gamma_1 & \frac{g}{S} \\ \gamma_1 - \gamma_2 - \frac{g}{S}\gamma_1\gamma_2 & 1 - \frac{g}{S}\gamma_2 \end{pmatrix}. \qquad (2.179)$$

Let us find the refractive indices of trajectory γ_1, γ_2. Regarding, at the entrance of the clearance

$$\gamma_1 = \frac{1}{x}\Delta\left(\frac{dx}{d\tau}\right).$$

Let us integrate equation of motion (2.53) along the period of reentry, bearing in mind that in the period of reentry the particle displacement from the axis does not manage to change

$$\Delta\frac{dx}{d\tau} = -x\int_{-\infty}^{-g/2S} Q(\tau)\,d\tau. \qquad (2.180)$$

In integral (2.180) the variable/alternating τ is counted off from the geometric center of clearance. Substituting in integral (2.180) function (2.57), we obtain

$$\gamma_1 = -\frac{eS}{2\varepsilon_0\beta^2\gamma^3}\int_{-\infty}^{-g/2}\frac{\partial E_z}{\partial z}\,dz,$$

whence

$$\gamma_1 = -\frac{eS}{2\varepsilon_0\beta^2\gamma^3}E_z\left(-\frac{g}{2}\right). \qquad (2.181)$$

For the second refractive index occurs the analogous expression

$$\gamma_2 = \frac{eS}{2\varepsilon_0\beta^2\gamma^3}E_z\left(\frac{g}{2}\right). \qquad (2.182)$$

If synchronous phase in the absolute value exceeds 8-10°, then it is possible to disregard the effect, connected with a change in the particle speed in the clearance. Then

$$E_z(z) = \frac{L}{g}E_0\cos\left(\frac{\omega z}{v} - \varphi\right), \qquad (2.183)$$

where E_0 - middle field; L - length of the period of the accelerating structure; φ - phase of standing wave at the moment of the time when particle passes the center of clearance. Let us substitute expression (2.183) into equalities (2.181), (2.182) and moreover for the factor of transit time (1.28) the approximation formula

$$T = \frac{\sin\frac{\pi g}{\beta\lambda}}{\frac{\pi g}{\beta\lambda}}. \qquad (2.184)$$

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Under the assumptions

$$\gamma_1 = \gamma \frac{L}{S} \cdot \frac{\cos\left(\varphi - \frac{\pi g}{\beta\lambda}\right)}{\sin \frac{\pi g}{\beta\lambda} \sin\varphi};$$

$$\gamma_2 = -\gamma \frac{L}{S} \cdot \frac{\cos\left(\varphi + \frac{\pi g}{\beta\lambda}\right)}{\sin \frac{\pi g}{\beta\lambda} \sin\varphi}.$$

indicated The parameter γ is determined by expressions (2.169), (2.170). Hence

$$\gamma_1 + \gamma_2 = 2\gamma \frac{L}{S}.$$

When $\left(\frac{\Omega}{\omega}\right)^2 \ll 1$ we have $\frac{g}{S}\gamma_{1,2} \ll 1$ and the matrix/die of accelerating clearance (2.179) is reduced to the form

$$\Gamma = \begin{pmatrix} 1 & \frac{g}{S} \\ 2\gamma \frac{L}{S} & 1 \end{pmatrix}. \qquad (2.185)$$

If the accelerating clearance sufficiently narrow, g/S<<1, then its action is reduced only to the refraction of the trajectory:

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 2\gamma \frac{L}{S} & 1 \end{pmatrix}. \qquad (2.186)$$

Ratio S/L - a number of accelerating clearances in one period of the focusing field. As it is easy to show,

$$\begin{pmatrix} 1 & 0 \\ 2\gamma \frac{L}{S} & 1 \end{pmatrix}^{S/L} = \begin{pmatrix} 1 & 0 \\ 2\gamma & 1 \end{pmatrix}.$$

Thus, and in the case of accelerating the beam in the field of standing waves the factor of defocusing (2.170) characterizes the complete defocusing of beam by high-frequency field in period S.

Let us note that at the length of accelerating gap function $Q_\omega(\tau)$ (2.57), entering the equations of motion (2.53), can be approximately replaced with average value, constant by the length of the gap/interval

$$\bar{Q}_\omega = \frac{S}{g} \int_{g/S} Q_\omega(\tau)\, d\tau. \qquad (2.187)$$

Then equations of motion in the section of accelerating gap will contain the constant coefficient:

$$\frac{d^2x}{d\tau^2} + \bar{Q}_\omega x = 0. \qquad (2.188)$$

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Actually/really, from formula (2.187) it follows

$$Q_\omega = -2\gamma \frac{L}{g}. \qquad (2.189)$$

so that the solution of equation (2.188) coincides with the result, obtained above. By equation with the averaged coefficient it is convenient to use when is not utilized matrix symbolism.

Into the product, which is determining the matrix/die of the period of focusing field, can enter matrices of several accelerating gaps. If the period of focusing field is short in comparison with the period of longitudinal vibrations, then in all matrices/dies of accelerating gaps can be used one and the same value/significance of the factor of defocusing.

§ 2.6. Selection of the parameters of the strong-focusing channel.

Let us examine several strong-focusing systems, on an example of which it will be possible to explain the characteristic laws, important for the identification of the parameters of the strong-focusing channels.

Let us assume that the strong-focusing channel continuously consists of the alternating lenses (channel FD, see Fig. 2.5). From the considerations of symmetry it is obvious that the maximum of Floquet's modulus/module falls in this channel accurately to the middle of the focusing lens, and the minimum - to the middle of the defocusing lens. Let us note that the channel FD is better to begin with the half lenses, since then the initial conditions for the matched beam are simplified: matched beam must have at the entrance of channel a crossover. Matching conditions will be satisfied, if we fit by correspondingly beam section in the crossover. Let us begin period from the middle of the focusing lens:

$$T = P_{1/2}\bar{P}P_{1/2}. \quad (2.190)$$

Matrix/die $P_{1/2}$ corresponds to half lens and, as it is easy to show, has the form

$$P_{1/2}=\begin{pmatrix} \cos\frac{K}{2} & \frac{D}{SK}\sin\frac{K}{2} \\ -\frac{SK}{D}\sin\frac{K}{2} & \cos\frac{K}{2} \end{pmatrix}. \qquad (2.191)$$

Let us expand/develop product (2.190):

$$\begin{aligned} &T_{11}=T_{22}=\operatorname{ch}K\cos K; \\ &T_{12}=\frac{D}{SK}(\operatorname{ch}K\sin K+\operatorname{sh}K); \qquad (2.192) \\ &T_{21}=-\frac{SK}{D}(\operatorname{ch}K\sin K-\operatorname{sh}K). \end{aligned}$$

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The matrix/die of period in the perpendicular plane

$$\overline{T}=\overline{P}_{1/2}P\overline{P}_{1/2}$$

can be obtained from expressions (2.192) by replacement of K by value iK:

$$\begin{aligned} &\overline{T}_{11}=\overline{T}_{22}=\operatorname{ch}K\cos K; \\ &\overline{T}=\frac{D}{SK}(\cos K\operatorname{sh}K+\sin K); \qquad (2.193) \\ &\overline{T}_{21}=\frac{SK}{D}(\cos K\operatorname{sh}K-\sin K). \end{aligned}$$

In the channel S=2D in question. The independent parameters of matrices/dies are determined by formulas (2.153). For both matrices/dies $\varepsilon=0$. The remaining parameters

$$\cos\mu=\operatorname{ch}K\cos K; \qquad (2.194)$$

$$\nu_{\Phi}=2K\sqrt{\frac{\operatorname{ch}K\sin K-\operatorname{sh}K}{\operatorname{ch}K\sin K+\operatorname{sh}K}}; \qquad (2.195)$$

$$\nu_{\text{Д}}=2K\sqrt{\frac{\sin K-\cos K\operatorname{sh}K}{\sin K+\cos K\operatorname{sh}K}}, \qquad (2.196)$$

where ν_{Φ} — dimensionless frequency in the middle of the focusing lens; v_{d} — the same in the middle of the defocusing lens. In accordance with condition (2.151) the particle trajectories are stable with

$$\operatorname{ch} K \cos K < 1. \qquad (2.197)$$

In Fig. 2.16 is constructed the curve according to formula (2.194). Along axis K is a series/row of the intervals, in which is satisfied the condition for stability (2.197). The first and widest stability region lies/rests within the limits

$$0 < K < 1{,}87. \qquad (2.198)$$

With an increase in the hardness of lenses the following stability regions rapidly decrease in the width. All stability regions after the first are arranged/located about zero functions cos K, i.e., about the points

$$K = (2n-1)\frac{\pi}{2}; \qquad n = 2, 3, \ldots$$

The width of stability region with n⩾2 can be calculated according to approximate formula

$$\Delta K = \frac{2}{\operatorname{ch}(2n-1)\frac{\pi}{2}}. \qquad (2.199)$$

From formula (2.199) it follows: ΔK=0.036 with n=2; ΔK=0.0016 with n=3.

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The field gradients of quadrupole lenses usually are selected so that the hardness of lenses it lay/rested at the first stability region (2.198). The use of the following stability regions requires too high a stability of gradient. Actually/really, according to expression (2.165)

$$\frac{\Delta K}{K}=\frac{1}{2}\frac{\Delta G}{G} \qquad (2.200)$$

it is easy to see that entire/all second stability region passes with a change in the gradient to 1.5o/o.

Is optimum value cos μ within the limits of the first stability region it corresponds to greatest possible value/significance ν_Φ, since in this case is provided maximum channel capacity. Let us construct the dependence of dimensionless frequencies ν_Φ, $\nu_\text{л}$ on value cos μ. These dependences in the parametric form are assigned by expressions (2.194)-(2.196). With cos μ=1 we have $K=0,\ \nu_\Phi=\nu_\text{л}=0$. With cos μ=-1

$$\cos K=-\frac{1}{\operatorname{ch} K};\quad \sin K=\operatorname{th} K.$$

Hence

$$\left.\begin{array}{l}\nu_\Phi\to 0\\ \nu_\text{л}\to\infty\end{array}\right\}\ \overset{(1)}{\text{при}}\ \cos\mu\to-1.$$

Key: (1). with.

The form of the function $\nu_\Phi(\cos\mu)$ and $\nu_\text{л}(\cos\mu)$ is shown in Fig. 2.17. Maximum ν_Φ, as follows from Fig. 2.17, it begins with cos μ=0.164, moreover at the point of maximum $\nu_\Phi=0.66$.

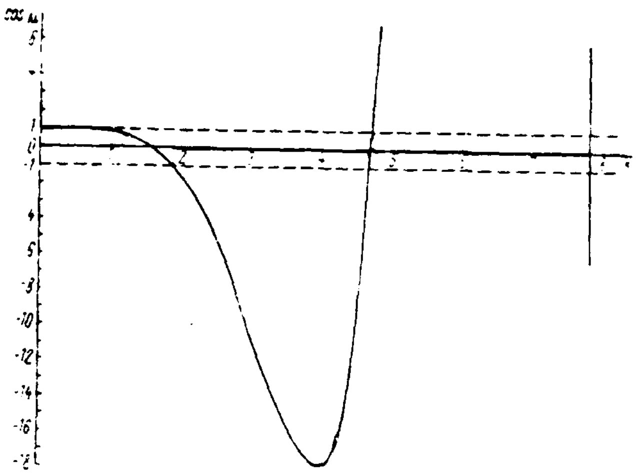

Fig. 2.16.

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However, maximum is flat and it is possible with unessential decrease v_Φ (to 5-8o/o) to increase cos μ to 0.4-0.5. This makes it possible to considerably lower the gradients of focusing fields, since with increase in cos μ the hardness of lenses and, consequently, also field gradients rapidly decrease.

The position of maximum v_Φ on axis cos μ depends on the structure of period, but in the majority of the cases this maximum proves to be in the interval 0.2< cos μ<0.4.

At given value ν_Φ (in other words, with the assigned hardness of lenses) it is possible to theoretically how conveniently raise the frequency of transverse vibrations $\omega_{r\,\text{мин}}$, by decreasing the length of period S=2D, since

$$\omega_{r\,\text{мин}} = \frac{v}{S}\nu_\Phi, \qquad (2.159a)$$

and in this case to respectively increase channel capacity. But with an increase in the frequency of transverse vibrations increases the gradient of the focusing field, since $K^2 \doteqdot D^2G$.

Before converting/transferring to the focusing periods of other types, let us examine the approximation/approach of "thin" lenses. If the length of the period of focusing field is fixed/recorded, then with the decrease of the length of lenses falls the value of hardness necessary for assigned cos μ. Decrease of hardness is connected with the fact that the idle gaps/intervals between the lenses amplify the effect, created by each lens. However, since the decrease of hardness is accompanied by the decrease of the length of lenses, the gradient of focusing field does not fall, but even somewhat increases. Thin lenses, for which D/S<<1, are commonly used in the ion guides. The strong-focusing optic/optics with thin lenses D/S<<1 can be used also in the linear accelerators to the superhigh energies.

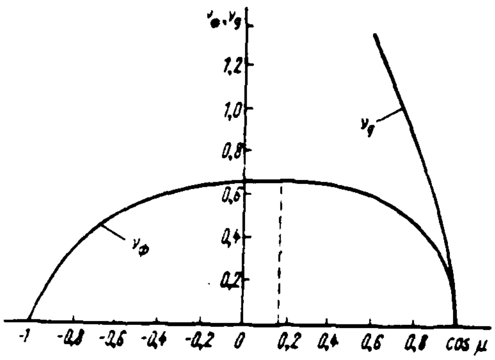

Fig. 2.17.

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However, with the energies in the range 0.5-100 MeV the parameters of the strong-focusing linear accelerators are usually such, that for the preliminary estimations it is possible successfully to consider quadrupole lenses "thin". Let us refine the concept of the "fineness" of lens. For this let us represent the matrix/die of lens (2.166) in the form of the product of three matrices/dies (2.174):

$$P=\begin{pmatrix} 1 & \frac{D}{SK}\operatorname{tg}\frac{K}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{SK}{D}\sin K & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{D}{SK}\operatorname{tg}\frac{K}{2} \\ 0 & 1 \end{pmatrix}. \quad (2.201)$$

Outer matrices/dies are equivalent to idle gaps/intervals with length along the axis z, equal to D/K tg K/2. Average/mean matrix/die corresponds to refraction of trajectory (jump of derivative) in each of principal planes of thick lens (Fig. 2.18). By analogy with optic/optics it is possible to introduce the concept of the focal distance of lens, calculated off principal plane II:

$$f = \frac{D}{K \sin K}. \quad (2.202)$$

Expression (2.202) follows from the simple geometric considerations, if we trace the particle trajectory, which was moving prior to the entrance into the lens in parallel to axis. Lens is called thin, if its length substantially less than the focal length. According to expression (2.202), the lens is thin, when its hardness is sufficiently small

$$K \sin K \approx K^2 \ll 1. \quad (2.203)$$

For satisfaction of the condition of "fineness" it is not compulsory so that the length of lens would be small in comparison with the length of the period of focusing field.

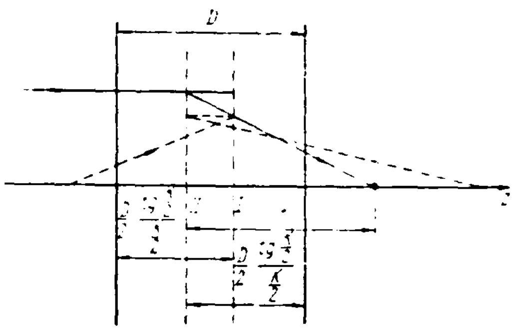

Fig. 2.18.

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But if lens is long, then its focal length must be large. Let us write the ratio of the length of lens to its focal length in the form:

$$\frac{D}{f} = \frac{D}{S} \cdot \frac{S}{f}.$$

Let us examine two different cases of satisfaction of the conditions of "fineness":

1. The length of lens is substantially lower than the length of the period of the focusing field, but the focal length of lens can be commensurated with the length of the period:

$$\frac{D}{S} \ll 1; \quad \frac{S}{f} < 1. \qquad (2.204)$$

2. Focal distance of lens is substantially more than the length of period of focusing field, but length of lens can be commensurated

with length of period:

$$\frac{S}{l} \ll 1;\ \frac{D}{S} \leqslant \frac{1}{2}. \qquad (2.205)$$

With satisfaction of the general condition of fineness (2.203) the matrix elements of lens it is possible to expand in power series on K and to use a finite number of terms of expansion. In the case (2.204) it proves to be sufficient to be restricted to the first terms of expansion. If D/S~1/2, then it is necessary to draw the second terms of expansion. But if condition (2.205) is satisfied, then equations of motion can be solved in the "smooth" approximation/approach which let us examine in § 2.7. In the smooth approximation/approach the ratio of the length of lens to the length of period can be any, provided the focal length of each lens was much more than period.

Under condition (2.204) the use/application of approximation/approach of thin lenses is especially simple. By replacing

$$\operatorname{tg}\frac{K}{2} \approx \frac{K}{2};\ \sin K \approx K,$$

we have

$$P = \begin{pmatrix} 1 & \frac{D}{2S} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{SK^2}{D} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{D}{2S} \\ 0 & 1 \end{pmatrix}. \qquad (2.206)$$

The matrix/die of thin lens is equivalent to two idle gaps/intervals, by length D/2, divided by one refracting plane.

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Since D/S<<1, matrix/die (2.206) even more is simplified

$$P=\begin{pmatrix} 1 & 0 \\ -\frac{SK^2}{D} & 1 \end{pmatrix} \qquad (2.206a)$$

and thin lens is replaced by refracting plane. For simplification in further recordings let us introduce designation

$$b=\frac{SK^2}{4D}. \qquad (2.207)$$

Then the matrices/dies of thin lens will take the form

$$P=\begin{pmatrix} 1 & 0 \\ -4b & 1 \end{pmatrix}; \quad \bar{P}=\begin{pmatrix} 1 & 0 \\ 4b & 1 \end{pmatrix}. \qquad (2.208)$$

It is easy to show that the same matrices/dies for thin lens can be obtained by another method, by assuming that a change in the particle displacement at the length of lens is negligible. Actually/really, by integrating equation of motion in the section of longitudinal axis, occupied with the field of lens (taking into account the real distribution of gradient along the axis), with x=const we obtain

$$\Delta\frac{dx}{d\tau}=-x\frac{eS^2}{p}\int G(\tau)\,d\tau. \qquad (2.209)$$

Hence

$$\Delta\frac{dx}{d\tau}=-4\frac{S}{4D}K^2x,$$

where

$$K^2=D^2\frac{e}{p}\cdot\frac{1}{D}\int G(z)\,dz. \qquad (2.210)$$

Comparing expressions (2.165) and (2.210), it is possible to see that K - hardness of thin lens, determined with the effective value of the gradient

$$G_{\text{эфф}} = \frac{1}{D} \int G(z)\,dz. \qquad (2.211)$$

Thus, if we determine the hardness of thin lens by the formula

$$K = D\sqrt{\frac{eG_{\text{эфф}}}{p}}. \qquad (2.212)$$

then

$$\Delta \frac{dx}{d\tau} = -4bx.$$

The length of lens, entering equality (2.211), can be selected arbitrarily, since initial expression (2.209) this length does not contain.

Let us examine channel FODO (see Fig. 2.6), which consists of thin lenses D/S<<1. The matrix/die of the period, which begins from the middle of the focusing lens, is equal to the product

$$T = P_{1/2} H \bar{P} H P_{1/2}.$$

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The length of idle gaps/intervals according to the condition is equal to S/2. We have

$$T = \begin{pmatrix} 1-2b^2 & 1+b \\ -4b^2(1-b) & 1-2b^2 \end{pmatrix}.$$

In matrix/die $\bar{T}$ reverses sign b. Then, according to expression (2.153),

$$\cos\mu = 1 - 2b^2;$$
$$\nu_\Phi = 2b\sqrt{\frac{1-b}{1+b}};\qquad (2.213)$$
$$\nu_\text{Д} = 2b\sqrt{\frac{1+b}{1-b}}.$$

To the beginning of the first stability region corresponds b=0; cos $\mu = 1$; $\nu_\Phi = \nu_\text{Д} = 0$. At the end of the first stability region b=1; $\cos\mu = -1$; $\nu_\Phi = 0$; $\nu_\text{Д} \to 0$. Derivative $\frac{d\nu_\Phi}{db}$ becomes zero with b=0.62. Maximum value/significance $\nu_\Phi = 0.60$ attains at cos μ=0.24. General/common/total variation ν_Φ, $\nu_\text{Д}$, on cos μ it is qualitative the same as in Fig. 2.17.

When the period of focusing field contains accelerating gaps, the selection of the medium frequency of transverse vibrations becomes complicated. Let us return to the period of the type FOD with the accelerating clearance whose matrix/die is determined by product (2.161). For simplification in the problem let us disregard/neglect the idle gaps/intervals between the lenses and we will consider the accelerating clearance infinite narrow (g/S<<1):

$$T = \overline{P}\Gamma P. \qquad (2.161a)$$

Matrix/die Γ is assigned by expression (2.186). In this case for the period falls one accelerating clearance, so that it is possible to assume L=S:

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 2\gamma & 1 \end{pmatrix}. \qquad (2.214)$$

Turning/running up the matrix/die of period, we obtain

$$\cos\mu = \operatorname{ch} K \cos K + \gamma \frac{\operatorname{ch} K \sin K + \operatorname{sh} K \cos K}{2K}. \tag{2.215}$$

The medium frequency of transverse vibrations can be selected, if to use the diagram of stability in the coordinates γ, ~~by~~ K, proposed by Smith and Glyukstern [32]. Let us apply on plane γ, K family of curves $K(\gamma)$, which correspond to different fixed values of cos μ (Fig. 2.19). Factor of defocusing (2.170) depends on the phase of particle.

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Operational conditions on the diagram of Smith-Glyukstern is usually selected in order to not exceed the limits of the stability region of transverse vibrations in any phases of the particles, which lie within separatrix (1.59). The region of stable phases lies/rests within the limits

$$2\varphi_s < \varphi < -\varphi_s. \tag{2.216}$$

Hence we obtain the limits of a change in the factor of the defocusing

$$-\gamma_s < \gamma < 2\gamma_s \cos\varphi_s. \tag{2.217}$$

For the stabilization of transverse vibrations of all particles, seized into acceleration mode, operating point should be selected inside the shaded rectangle which is limited by the curves cos μ=1 and cos μ=-1 (see Fig. 2.19). The limits of the permissible change in value cos μ for the synchronous particle are determined by the cut of vertical straight/direct with abscissa $\gamma=\gamma_s$, lying in limits

$$\cos\mu_1 < \cos\mu_s < \cos\mu_2.$$

For an increase in the channel capacity value cos μ should be selected in these limits so as to ensure maximally high value/significance $\nu_{\Phi}=\nu_{\text{мин}}$.

For the preliminary estimations it is possible not to resort to the diagram of Smith-Glyukstern, construction of which requires, as a rule, cumbersome calculations, but using the simplified formula for cos μ. Let us represent the equality (2.215) in the form

$$\cos\mu = \cos\mu_{\Phi} - \gamma f(K),$$

where

$$\cos\mu_{\Phi} = \operatorname{ch} K \cos K$$

characterizes the focusing channel in the absence of accelerating gaps, and factor with the factor of defocusing is equal to

$$f(K) = \frac{1}{2K}(\operatorname{ch} K \sin K + \operatorname{sh} K \cos K).$$

This factor noncritically depends on K, after remaining close to unity over wide limits of hardness change:

$$f(K) \approx 1 - \frac{K^4}{30} + \frac{K^8}{22\,680} - \ldots$$

In interval $0<\cos\mu_{\Phi}<1$ factor f differs from unity less than to 20o/o and in the approximate computations it can be replaced by one; then we obtain the simplified formula

$$\cos\mu = \cos\mu_{\Phi} + \gamma. \qquad (2.218)$$

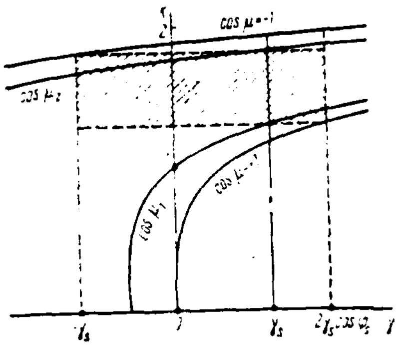

Fig. 2.19.

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Formula (2.218) is approximately valid not only for the period in question. Subsequently let us show that in the smooth approximation/approach formula (2.218) occurs in the general case.

Formulas (2.214), (2.215) relate to one of the simplest periods of focusing field in the linear accelerators. In the more complicated cases the period contains several lenses of one sign and several accelerating gaps. Into the product, which is determining the matrix/die of period, enter as many matrices/dies P and with respect to $\overline{P}$, as lenses of one sign in the period (n), matrices/dies Γ and as accelerating gaps and period (S/L). Table 2.1 gives values n and S/L for some proton accelerators. In the general case value cos μ -

the complex function of the parameters, which are determining a number of identical matrices/dies in the product and elements/cells of these matrices/dies in different sections of period. Let us disregard/neglect idle gaps/intervals. Then such parameters prove to be six:

$$\cos\mu = f\left(K, \gamma\frac{L}{S}, \frac{D}{S}, \frac{g}{S}, \frac{S}{L}, n\right). \qquad (2.219)$$

The diagram of Smith-Glyukstern it is necessary to construct each time anew for the fixed values of four latter/last parameters in expression (2.219). However, for the preliminary estimations during the selection of the structure of period problem can be substantially simplified, if to introduce two assumptions:

1. The length of each accelerating gap is much lower than the length of the period of the focusing field

$$\frac{g}{S} \ll 1; \ \frac{S}{nD} = 2.$$

Table 2.1.

(1) Ускорители	(2) Тип периода	(3) Число ускоряющих промежутков S/L	(4) Число линз одного знака n
(5) 1-й резонатор линейного ускорителя И-2	ФОД	1	1
(5a) 2-й резонатор линейного ускорителя И-2	ФОФДОД	2	2
(6) 1-й резонатор инжектора ЦЕРН Ускоритель И-100	ФОДО	2	1
(7) 2-й и 3-й резонаторы инжектора ЦЕРН	ФОФОДОДО	4	2

Key: (1). Accelerators. (2). Type of period. (3). Number of accelerating gaps. (4). Number of lenses of one sign. (5). 1st resonator of linear accelerator I-2. (5a). the 2nd resonator of the linear accelerator I-2. (6). 1st resonator of injector CERN accelerator I-100. (7), 2nd and 3rd resonators of injector CERN.

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2. Defocusing action of all accelerating gaps in one period can be noticed by their cumulative effect, led to one point of period. . Then, for example,

$$T=\Gamma\overline{P}\Gamma\overline{P}\Gamma P\Gamma P\approx\overline{P}\overline{P}\Gamma\Gamma\Gamma\Gamma PP=\overline{P}^2\Gamma^4P^2.$$

In the general case we will obtain the matrix/die of period, analogous(2.161a),

$$T\approx\overline{P}^n\Gamma^{S/L}P^n. \qquad (2.220)$$

But matrix/die P_{SL} exists (2.214), and raising to the power of matrix of lens analogous (with 2.178) gives

$$P^n_{\,} = \begin{pmatrix} \cos K & \frac{1}{2nK}\sin K \\ -2nK\sin K & \cos K \end{pmatrix}^n = \begin{pmatrix} \cos nK & \frac{1}{2nK}\sin nK \\ -2nK\sin nK & \cos nK \end{pmatrix}.$$

so that

$$\cos\mu = f(nK, \gamma),$$

where f - function (2.215), in which hardness K is replaced by value nK. Thus, the introduced above simplifying assumptions make it possible to construct the diagram of Smith-Glyukstern in the coordinates γ, nK, approximately valid for any values of n, S/L. This diagram coincides with the diagram, given in Fig. 2.19, if we replace the designation of axis K by nK.

Field gradient in the quadrupole lenses is connected with parameter nK with the relationship/ratio

$$nK = nD\sqrt{\frac{eG}{p}},$$

where nD - total length of the lenses of one sign in the period. It is increased a number of lenses of one sign in the period, keeping constant the length of lenses and after increasing respectively the length of period. Then

1) with assigned cos μ is retained nK and, therefore, field gradient in the lenses falls as $1/n^2$;

2) with an increase in length of period S~~2nD increases the factor of defocusing of synchronous particle (2.169) and, therefore, the spread/scope of oscillations γ increases as n^2. The region of allowed values cos μ is compressed and with sufficiently high values γ_s becomes zero;

3) with an increase in the length of period decreases the minimum frequency of transverse vibrations (2.159). Channel capacity falls as 1/n.

Consequently, an increase of the number of lenses of one sign in the period (with D=const) makes it possible to substantially lower the power, spent on the focusing, but in this case is decreased stability region and descends channel capacity.

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In the initial part of the proton linear accelerator with the energies of injection to 1 MeV system n=2 does not provide the stability of particles with all phases and limits of separatrix. In further parts of the accelerator when the factor of defocusing of synchronous particle already sufficiently attenuates, this system can be used, if is admissible a reduction in the capacity.

For the analysis of the focusing channels with the accelerating gaps it is possible to utilize another diagram, sometimes more convenient. This diagram let us examine subsequently.

Since the medium frequency of transverse vibrations for synchronous particle μ_s is selected so as to ensure the stability of all particles, seized into acceleration mode, and to obtain a maximally high value/significance of capacity, then is expedient to support $\cos\mu_s$ with constant at the length of accelerator. This condition does not always succeed in maintaining/withstanding, since in the beginning of accelerator the gradient of focusing field can prove to be too great and virtually unrealizable. In that case is begun the channel from the lower value/significance of medium frequency, and then they gradually lead medium frequency to the optimum value. In view of the fact that the phase volume of beam (2.125) remains invariant, the semi-axes of beam with an adiabatic increase in the frequency decrease proportionally $\frac{1}{\sqrt{\omega_r}}$.

§ 2.7. Floquet's functions in the smooth approximation/approach. Parametric resonances.

The matrix method of calculating Floquet's functions gives solution at one chosen point of period. This method is convenient for determining the matching conditions, calculating the medium frequency of transverse vibrations, evaluation of stability region. At the same time method possesses deficiencies/lacks. For example, the volume of computational work rapidly grows/rises with an increase of the number of different sections in the period.

Smooth approximation/approach gives solution for Floquet's functions directly depending on τ and does not require the approximation of fields in the quadrupole lenses by square waves. The approximate smooth solution proves to be sufficiently to precise ones, if phase change of transverse vibrations satisfies the condition

$$\frac{\mu}{2\pi} \ll 1. \qquad (2.221)$$

Condition (2.221) means that the period of transverse vibrations must be considerably more than the period of focusing field. Smooth approximation/approach is reduced actually to the first approximation of the method of the "averaging", developed by N. N. Bogolyubov and by Yu. A. Mitropol'skiy [69]. Generally, smooth approximation/approach gives sufficiently accurate results in all cases when the frequency of periodic effect is much higher than the natural frequency of dynamic system [70, 71, 21].

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We will search for the solution of equation (2.62) in the following form:

$$x(\tau)=[1+q(\tau)]X(\tau). \quad (2.222)$$

Let us substitute solution (2.222) in equation (2.62):

$$\frac{d^2X}{d\tau^2}+2\frac{dq}{d\tau}\cdot\frac{dX}{d\tau}+q\frac{d^2X}{d\tau^2}+\frac{d^2q}{d\tau^2}X-QX-qQX=0. \quad (2.223)$$

Further, let us determine function $q(\tau)$ with the aid of the differential equation

$$\frac{d^2q}{d\tau^2}=-Q(\tau)+\overline{Q}, \quad (2.224)$$

where $\overline{Q}$ - average/mean value/significance of function $Q(\tau)$ in the period

$$\overline{Q}=\int_{\tau}^{\tau+1}Q(\tau)\,d\tau. \quad (2.225)$$

First-order derivative $\frac{dq}{d\tau}(\tau)$, just as the second, periodic functions τ with the period $\Delta\tau=1$. Actually/really, from expression (2.224) it follows

$$\frac{dq}{d\tau}(\tau+1)-\frac{dq}{d\tau}(\tau)=\int_{\tau}^{\tau+1}\frac{d^2q}{d\tau^2}\,d\tau=0.$$

Function $q(\tau)$ is assigned with the aid of the differential second order equation and contains two arbitrary constants. Let us select one of the constants in such a way that would be satisfied the condition

$$\overline{\frac{dq}{d\tau}} = \int_{\tau}^{\tau+1} \frac{dq}{d\tau}(\tau)d\tau = 0. \qquad (2.226)$$

Then original q(τ) will also be periodic function, since

$$q(\tau+1) - q(\tau) = \int_{\tau}^{\tau+1} \frac{dq}{d\tau} d\tau = 0.$$

The second arbitrary constant let us determine by the condition

$$\bar{q} = \int_{\tau}^{\tau+1} q(\tau)\, d\tau = 0. \qquad (2.227)$$

Thus, equation (2.224) together with conditions (2.226), (2.227) uniquely determines periodic function with the period $\Delta\tau$=1 and with the constant component, the the equal to zero.

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If the distribution of gradient along the axis of channel can be approximated by the sinusoid

$$Q(\tau)=\frac{S^2K^2_{\text{макс}}}{D^2}\cos 2\pi\tau,$$

then function q - the harmonic:

$$q(\tau)=\frac{S^2K^2_{\text{макс}}}{4\pi^2D^2}\cos 2\pi\tau.$$

In the general case function q(τ) is always substantially smoother, than the coefficient of the equation of Mathieu-Hill. Actually/really, function q - the result of the dual integration of this coefficient. For an example Fig. 2.20 gives the course of coefficient Q(τ) in the channel FD (with the approximation of fields by square waves) and corresponding to this coefficient function q(τ).

Substituting expression (2.224) in equation (2.223), we obtain

$$(1+q)\frac{d^2X}{d\tau^2}+(qQ+\overline{Q})X+$$
$$+2\frac{dq}{d\tau}\cdot\frac{dX}{d\tau}=0. \quad (2.223a)$$

From equation (2.223a) in the principle is determined function X(τ) for the exact solution. however, equation (2.223a) is not simpler than the initial, since its coefficients are also the periodic functions of time. Therefore we will search for approximate solution of equation (2.23a), assuming that function $X(\tau)$ is changed slowly, so that its change in each period of focusing field can be disregarded/neglected. Then in accordance with expression (2.227)

$$\overline{x(\tau)} = X(\tau) - \overline{q(\tau)}\, X(\tau) \approx X(\tau).$$

Value X in expression (2.222) - this is the average/mean value/significance of trajectory in each period of focusing field. It is averaged for the period equation (2.223a). Utilizing conditions (2.226), (2.227), we obtain

$$\frac{d^2X}{d\tau^2} - \mu^2 X = 0. \qquad (2.228)$$

Into equation (2.228) is introduced the designation

$$\mu^2 = \overline{Q} + \int\limits_{\tau}^{\tau+1} q(\tau)\, Q(\tau)\, d\tau. \qquad (2.229)$$

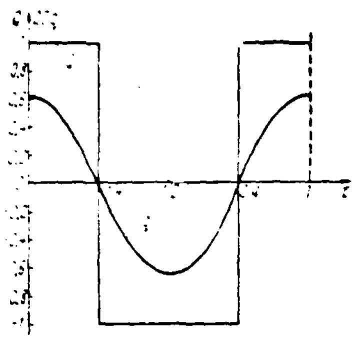

Fig. 2.20.

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The solution of equation (2.228) exists

$$X(\tau) = A\sin(\mu\tau - \theta). \quad (2.230)$$

Consequently,

$$x(\tau) = A[1 + q(\tau)]\sin(\mu\tau - \theta). \quad (2.231)$$

Value μ - the angular frequency of the fluctuations of the slow component of trajectory (2.222). Cyclic period of a function $q(\tau)$ is 2π. Slowness condition of function $X(\tau)$, obviously, is satisfied with $\mu \ll 2\pi$. Solution (2.231) in this case is the slow sinusoidal function, to which are superimposed "rapid" pulsations with the period of the focusing field (see Fig. 2.5).

Let us rate/estimate the spread/scope of oscillations of a function $q(\tau)$; it is close to the harmonic. Therefore

$$\frac{dq}{d\tau} \approx 2\pi q; \ \frac{d^2q}{d\tau^2} \approx -4\pi^2 q.$$

or

$$Q(\tau) \approx \overline{Q} - 4\pi^2 q(\tau).$$

Substituting this expression into formula (2.229), we have

$$\mu^2 \approx \overline{Q} - 4\pi^2 \overline{q^2}.$$

In the strong-focusing channel without the accelerating gaps the average/mean value/significance of the coefficient of the equation of Mathieu-Hill is always equal to zero. Let us find connection/communication between the factor of defocusing and the average/mean value/significance of this coefficient. Introducing function (2.55) into integral (2.225) and taking into account that number of accelerating gaps in the period is equal S/L, we obtain

$$\overline{Q} = \frac{S}{L} \int Q_\omega(\tau)\, d\tau.$$

According to expressions (2.187), (2.189),

$$\overline{Q} = -2\gamma. \qquad (2.232)$$

The mean square of function $q(\tau)$ proves to be approximately equal to

$$\overline{q^2} \approx \left(\frac{\mu}{2\pi}\right)^2 + \frac{\gamma}{2\pi^2}.$$

From conditions (2.221) and $\gamma \ll 1$ it follows that

$$\overline{q^2} \ll 1; \quad |q| \ll 1. \qquad (2.233)$$

Thus, rapid pulsations are small in comparison with the amplitude of the slow component of solution.

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The amplitude of approximate solution (2.231) - periodic function of the independent variable with the period $\Delta\tau$=1 and, therefore, is proportional to the modulus/module of Floquet's function

$$\varrho(\tau) \approx C[1+q(\tau)].$$

The phase of Floquet's function in the approximation/approach accepted exists

$$\Phi(\tau) \approx \mu\tau.$$

Value μ is equal to phase change of Floquet's function in the period $\Delta\tau$=1, which coincides with the determination μ, given in § 2.3. The condition for the standardization of fundamental solutions (2.65) leads to relationship/ratio (2.105), whence it follows

$$\mu = \frac{1}{C^2}\int_0^1 \frac{d\tau}{(1+q)^2}.$$

Let us expand integrand in series/row according to degrees of q and we will be restricted to linear term. Condition (2.227) gives

$$C = \frac{1}{\sqrt{\mu}}.$$

Approximation for the modulus/module of the normalized function of Floquet takes the form

$$\varrho(\tau) \approx \frac{1}{\sqrt{\mu}}[1+q(\tau)]. \qquad (2.234)$$

The instantaneous frequency of transverse vibrations (2.103) is equal to

$$\omega_r(\tau) \approx \frac{\mu}{T_\Phi (1+q)^2}, \qquad (2.235)$$

the envelope of matched beam (2.122)

$$r(\tau) \approx \sqrt{\frac{F_0}{\mu}}\,[1+q(\tau)]. \qquad (2.236)$$

The semi-axes of the section of matched beam in both transverse planes on the average are identical and equal to

$$R_c^0 = \sqrt{\frac{F_0}{\mu}}\,. \qquad (2.237)$$

Envelope is relatively weakly modulated with the period of focusing field, moreover the depth of modulation is uniquely determined by function q(τ). In the smooth approximation/approach the depth of modulation of envelope does not depend on the phase volume of beam.

Let us replace in integral (2.229) function Q(τ) with its expression from equation (2.224) and let us take integral in parts. As a result we have

$$\mu^2 = \bar{Q} + \int_0^1 \left(\frac{dq}{d\tau}\right)^2 d\tau. \qquad (2.238)$$

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Taking into account that coefficient Q(τ) - the sum of two functions (2.56) and (2.57), function q(τ) also can be divided into two components/terms/addends

$$q = q_\Phi + q_\omega$$

as follows

$$\frac{d^2 q_\Phi}{d\tau^2} = -Q_\Phi(\tau);$$

$$\frac{d^2 q_\omega}{d\tau^2} = -Q_\omega(\tau) + \bar{Q}_\omega.$$

Expression (2.238) will take the form

$$\mu^2 = -2\gamma + \int_0^1 \left[\left(\frac{dq_\Phi}{d\tau} \right)^2 + 2\, \frac{dq_\Phi}{d\tau} \cdot \frac{dq_\omega}{d\tau} - \left(\frac{dq_\omega}{d\tau} \right)^2 \right] d\tau.$$

Let us examine each component/term/addend under the integral. Value

$$\mu_\Phi^2 = \int_0^1 \left(\frac{dq_\Phi}{d\tau} \right)^2 d\tau$$

is a square of medium frequency in the channel without accelerating field. Further

$$\int_0^1 \frac{dq_\Phi}{d\tau} \cdot \frac{dq_\omega}{d\tau}\, d\tau = 0.$$

Actually/really, reference point τ always can be selected so that the function $Q_\omega(\tau)$ would be even, and function $Q_\Phi(\tau)$ - odd. Then $\frac{dq_\Phi}{d\tau}$ will be even function, and $\frac{dq_\omega}{d\tau}$ - odd. Integral on the period of the product of even and odd functions is equal to zero. The third integral is proportional $\gamma^2 \ll \gamma$ and it can be disregarded/neglected. Finally we will obtain

$$\mu^2 = \mu_\Phi^2 - 2\gamma. \tag{2.239}$$

If μ is small, then

$$\cos\mu \approx 1 - \frac{1}{2}\mu^2$$

and equality (2.239) can be rewritten in the form

$$\cos\mu = \cos\mu_\Phi + \gamma. \tag{2.218a}$$

This formula, obtained here in the smooth approximation/approach for any focusing structures, is valid between very wide limits of change μ_Φ. From the results of § 2.6 it is evident that for the rough estimates it it is possible to use up to negative values $\cos\mu_\Phi$, i.e. formula (2.218a) emerges far beyond the limits of the applicability of smooth approximation/approach.

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The maximum spread/scope of function $q(\tau)$ is proportional to the ratio of the length of period to the focal length of lenses. Thus, for the structure in Fig. 2.20 of equation (2.224) and conditions (2.226), (2.227) it follows

$$q_{макс}=\frac{Q}{32},$$

where Q - the fixed level of function $Q(\tau)$ in each lens. But from expressions (2.202) and (2.56), (2.165) we have

$$\frac{S}{f}\approx\frac{S}{D}K^2;\quad K^2=\left(\frac{D}{S}\right)^2 Q. \qquad (2.240)$$

Hence

$$q_{макс}\approx\frac{1}{16}\frac{S}{f}.$$

Thus, the spread/scope of oscillations $q(\tau)$ is substantially lower than the ratio S/f. Smooth approximation/approach is correct, when this relation little and in this sense it is one of limiting cases of approaching thin lenses (2.205). However, smooth approximation/approach does not require so that the ratio S/f would be very little, and therefore it gives sufficiently accurate results for the thick lenses when matrices/dies (2.208) prove to be already inapplicable.

The comparison of formulas (2.229), (2.234), (2.235) with the exact solutions, obtained as a result of the numerical integration of the equation of Mathieu-Hill, shows that smooth solution gives a good approximation/approach to μ (with the error within limits of 5-8o/o) in the interval 0.3<cos μ<1, i.e., in the entire virtually interesting part of the first stability region. Approximations/approaches to instantaneous frequencies prove to be more badly: accuracy indicated above are retained in the interval to cos μ=0.6-0.7.

The frequency of transverse vibrations depends on the phase of particle. In the smooth approximation/approach, according to expressions (2.239), (2.170),

$$\mu^2=\mu_\Phi^2-2\gamma_s\frac{\sin\varphi}{\sin\varphi_s}. \qquad (2.241)$$

The phase of the particle, seized into acceleration mode, varies around the synchronous phase and the medium frequency of transverse vibrations periodically is changed with the frequency of longitudinal vibrations. Longitudinal and transverse vibrations prove to be parametrically connected that it is possible to lead to the supplementary driving of transverse vibrations at frequencies of parametric resonance. By hypothesis, accepted above, the amplitude of the longitudinal component of accelerating field does not depend on transverse coordinates. Thereby it is disregarded by the parametric effect of transverse vibrations on the longitudinal ones.

Smooth approximation/approach makes it possible to simplify the problem about the parametric resonances of transverse vibrations.

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Let us assume that the phase of particle in the period of focusing field is changed negligibly little, but let us consider the change in the phase, connected with the longitudinal vibrations, at the entire length of accelerator. Let us examine only small longitudinal vibrations, for which in view of (1.47)

$$\frac{\sin\varphi}{\sin\varphi_s} \approx 1 + \psi \operatorname{ctg} \varphi_s. \qquad (2.242)$$

Equation (2.228) is reduced to the equation of Mathieu, since

$$\psi = \Phi \sin \Omega t, \qquad (2.243)$$

where Φ, Ω - respectively instantaneous values of amplitude and frequency of small of phase. From the expression for the factor of defocusing of synchronous particle (2.169) we have

$$(\Omega T_\Phi)^2 = 4\gamma_s. \qquad (2.244)$$

In the approximation/approach of small oscillations (2.242) the square of medium frequency (2.241) is equal to

$$\mu^2 = \mu_s^2 - 2\gamma_s \psi \operatorname{ctg} \varphi_s, \qquad (2.245)$$

where

$$\mu_s^2 = \mu_\Phi^2 - 2\gamma_s \qquad (2.246)$$

and μ_s - medium frequency of transverse vibrations of synchronous particle. Let us substitute expressions (2.243-2.245) in the equation for the slow component of transverse vibrations (2.228):

$$\frac{d^2 X}{d\tau^2} + (\mu_s^2 - 2\gamma_s \Phi \operatorname{ctg} \varphi_s \sin 2\sqrt{\gamma_s}\,\tau) X = 0. \qquad (2.247)$$

By the transformation of the independent variable

$$\xi = \frac{1}{\pi}\sqrt{\gamma_s}\,\tau$$

equation (2.247) is led to canonical form (2.108) with the values of the coefficients

$$a = \frac{\mu_s^2}{\gamma_s}; \quad q = \Phi \operatorname{ctg} \varphi_s. \qquad (2.248)$$

The ranges of change in coefficients (2.248), which correspond to the unstable solutions of the equation of Mathieu, are given in Fig. 2.11. The more detailed information about the character of the solutions of the equation of Mathieu is contained in book [59]. Taking into account relationships/ratios (2.106), (2.244), we have

$$a = \left(\frac{2\Omega_{rs}}{\Omega}\right)^2, \qquad (2.249)$$

where Ω_{rs} - the medium frequency of transverse vibrations of synchronous particle to scale of time t; Ω - frequency of small longitudinal vibrations.

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Parametric resonance takes place when the medium frequency of transverse vibrations of synchronous particle is close to half integral value/significance of the frequency of the small longitudinal vibrations:

$$\Omega_{rs} = \frac{n}{2}\Omega \qquad (n = 1, 2, 3, \ldots). \qquad (2.250)$$

In other words, the regions of parametric resonances appear about the values

$$\mu_s^2 = n^2\gamma_s,$$

or

$$\cos\mu_s \approx 1 - \frac{1}{2}n^2\gamma_s. \qquad (2.251)$$

According to expression (2.95), the excitation of parametric resonance occurs with the specific phase relationships/ratios between the transverse and longitudinal vibrations. Therefore parametric excitation they can test/experience not all particles. In the most unfavorable case from expression (2.95) it follows

$$X(\zeta) = Ae^{k\zeta}\rho(\zeta)\cos\Phi(\zeta). \qquad (2.252)$$

Parameter k for different regions of resonance can be calculated by the methods, presented, for example, in the book of MacLachlan [59]. The characteristic index of MacLachlan μ is connected with parameter k with relationship/ratio $k=\mu\pi$. If we rewrite solution (2.252) in the form

$$X(t) = Ae^{t/\Theta}\rho(t)\cos\Phi(t), \qquad (2.252a)$$

then the time constant of oscillation buildup will be equal to

$$\Theta = \frac{T_c}{k}, \qquad (2.253)$$

where $T_c = \frac{2\pi}{\Omega}$ - period of longitudinal vibrations. Width of band of resonance and time constant oscillation buildup in the middle of band depend on coefficient of q (2.248). Let us accept for further estimations $\Phi = |\varphi_s|$. Then when $\cos\varphi_s = 0{,}8$ we have q=0.86.

Let us give appropriate data for this value/significance of q.

1. First region of parametric resonance.

Width of unstable region

$$1-0.84\gamma_s<\cos\mu_s<1-0.045\gamma_s. \qquad (2.254)$$

Middle of the band of resonance (a=1)

$$\cos\mu_s=1-0.5\gamma_s.$$

Time constant of oscillation buildup in the middle of the band

$$\Theta=0.64T_c \qquad (2.255)$$

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2. Second region of parametric resonance. Width of unstable region

$$1-2.20\gamma_s<\cos\mu_s<1-1.96\gamma_s. \qquad (2.256)$$

Middle of the band of resonance (a=4)

$$\cos\mu_s=1-2\gamma_s.$$

Time constant of oscillation buildup in the middle of the band

$$\Theta=6.4T_c. \qquad (2.257)$$

The regions of parametric resonance are conveniently examined on plane $\gamma_s, \cos\mu_s$. If we apply to the same plane of the field of stable trajectories, then diagram in coordinates $\gamma_s, \cos\mu_s$ will make it possible to determine the optimum value of medium frequency taking into account parametric resonances. Let us establish for each value/significance γ_s the permissible range of change cos μ_s so that all particles, seized into acceleration mode, would have the stable trajectories of transverse vibrations in any phases in the limits of separatrix (2.216). From equalities (2.218a) and (2.170) it follows

$$\cos\mu = \cos\mu_s - \gamma_s\left(\frac{\sin\varphi}{\sin\varphi_s} - 1\right). \qquad (2.258)$$

When $\varphi = 2\varphi_s$

$$\cos\mu' = \cos\mu_s - \gamma_s(2\cos\varphi_s - 1)$$

When $\varphi = -\varphi_s$

$$\cos\mu'' = \cos\mu_s - 2\gamma_s.$$

Let us require satisfaction of the conditions

$$\cos\mu' < 1; \quad \cos\mu'' > 1.$$

From these two inequalities we determine the permissible boundaries of selection $\cos\mu_s$:

$$-(1-2\gamma_s) < \cos\mu_s < 1 - \gamma_s(2\cos\varphi_s - 1). \qquad (2.259)$$

In Fig. 2.21 is constructed the diagram of stability in coordinates γ_s, $\cos\mu_s$. The boundaries of the region of allowed values $\cos\mu_s$ are given by heavy lines. If the values of parameters γ_s and $\cos\mu_s$ are selected within the region, then by this are provided the conditions for lateral stability of all particles in the limits of separatrix. Upper boundary of the region depends on the selected synchronous phase. The point of intersection of boundaries has coordinates

$$\gamma_s = \frac{2}{1+2\cos\varphi_s}; \quad \cos\mu_s = \frac{3-2\cos\varphi_s}{1+2\cos\varphi_s}.$$

Lower boundary intersects the axis of abscissas at point $\gamma_s = 0.5$.

Stability limits on the diagram Fig. 2.21 are obtained on the basis of approximation formula (2.218a).

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Precise calculations show that stability limit somewhat wider. Lower boundary, obtained from precise formulas, lies/rests below than on Fig. 2.21, and intersects the axis of abscissas at point $\gamma_s \approx 0.65$. Thus, construction of diagram according to the formulas of smooth approximation/approach gives supply on the stability.

On the diagram Fig. 2.21 are shaded the regions of parametric resonances.

Diagrams of this type can be utilized instead of the diagrams of Smith-Glyukstern for the preliminary identification of the parameters of the strong-focusing linear accelerator. The parameters of accelerator are selected according to diagram 2.21 in the following sequence. First are established/installed the values of synchronous phase and specific acceleration, proceeding from the considerations, connected with dielectric strength of accelerating gaps, the permitted by expenditure high-frequency power and value of capture region. Subsequently these values can be more precisely formulated during the selection of other parameters. Strong focusing does not usually limit the selection of synchronous phase. Then are determined the frequency of small longitudinal vibrations and the factor of defocusing of synchronous particle (2.169) at the entrance of accelerator. During the determination of the input value/significance of the factor of defocusing is selected the permissible ratio $S/\beta\lambda$, which ensures value γ_{s} solved by diagram. By obtained value/significance γ_{s} in the limits, solved by diagram, is selected value $\cos\mu_{0}$ making it possible to obtain maximally high frequency $\nu_{\text{мин}}$ for the synchronous particle. Then they calculate the hardness of lenses from the matrix/die of period or (in the permissible cases) according to the formulas of smooth approximation/approach. By the assigned hardness is determined the law of a change in the gradients of lenses along the axis of accelerator, on the basis of the lengths known for each period of lenses and particle speed.

Let us examine an example. Let us assume for the linear accelerator of protons is selected the parameters: $\cos\varphi_{s} = 0.8$; $W_{\lambda} = 2.7\cdot10^{-3}$; energy of injection W_0=700 keV. At the entrance of accelerator $(\Omega/\omega)^2$=$8.73\cdot10^{-3}$. If we select $S/\beta\lambda$=4, then $\gamma_{s} = 1.38$ knowingly lies/rests at the unstable region.

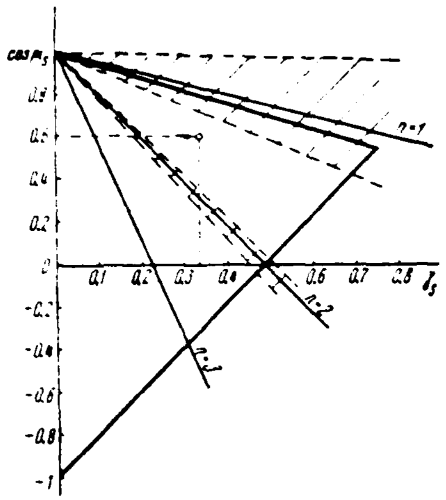

Fig. 2.21.

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With $S/\beta\lambda=2$ the factor of defocusing of synchronous particle is located already in the stable region the diagram Fig. 2.21. The time constant of the increase of the amplitude of transverse vibrations in the region of first parametric resonance (n=1) is equal, according to expression (2.255), approximately/exemplarily to half of the period of longitudinal vibrations. This region is dangerous, and during the selection of value $\cos\mu_s$ it should be avoided. Most adequate/approaching values $\cos\mu_s$ both for reasons, connected with the expenditure of power, and for reasons, connected with the channel capacity, is $\cos\mu_s \approx 0.45 - 0.60$. If value $\cos\mu_s$ is kept constant along the axis of accelerator, then due to the adiabatic fading of the factor of defocusing operating point on the diagram will be displaced to the left and beam will pass the regions the second, third and subsequent resonances. These regions are not already virtually terrible, since they pass rapidly and correspond to the slow responses of oscillation buildup. In the second region parametric resonance leads to the maximum building up of oscillations, knowingly less than 10o/o.

§ 2.8. Transverse vibrations in the imperfect strong-focusing system.

Until now, were examined the strong-focusing channels without the errors. It was assumed that the structural/design and electrical characteristics of all periods were identical. In the regular channel can be spread the matched beam. Different irregularities lead to the disagreements/mismatches of beam and, therefore, to an increase in its effective phase volume. Let us examine allowances for the parameters of accelerator, connected with transverse vibrations. For this let us calculate the disturbed (mismatched) parameters of beam after each irregularity and will determine the sizes/dimensions of the mismatched beam in the channel. In the general case the problem is reduced to the determination of Floquet's ellipse, circumscribed around the mismatched phase volume, i.e., to the calculation of coefficient ξ_1 [see expression (2.140)]. Angle of the slope α and relation of the semi-axes of Floquet's ellipse (see Fig. 2.12) are uniquely determined by the parameters of channel at each point of period. Above we connected values M/N and α with the modulus/module of Floquet and his derivative (2.119a). For the geometric constructions it is useful to express the relation of semi-axes and the angle of the slope of Floquet's ellipse through the matrix elements of period. The substitution of expressions (2.160) into formulas (2.119a) gives

$$\frac{M}{N}=\frac{1}{2\nu\cos\varepsilon}\left[1+\nu^2+\sqrt{(1-\nu^2)^2-4\nu^2\cos^2\varepsilon}\right],$$

$$\operatorname{tg}2\alpha=-\frac{2\nu\sin\varepsilon}{1-\nu^2}. \qquad (2.260)$$

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At the points of the extremum of Floquet's modulus/module $\varepsilon=0$ and $\frac{N}{M}=\nu;\ \alpha=0$. Let us examine the basic forms of the errors which can arise in the strong-focusing channel of linear accelerator. The given cases show the possible procedure of calculation of allowances.

a. Idle gap in channel. Let us assume that two parts of the focusing channel are divided by the idle gap/interval with a length of 1, which disrupts the regularity of channel. This gap/interval, in particular, can arise between the adjacent resonators of multicavity accelerator. If in the first resonator beam was matched with the channel, then after the passage of idle gap/interval due to the drift of particles along the axis x with $dx/d\tau$=const beam will be mismatched and its envelope will cease to be periodic function. The effective phase volume of beam in this case grows/rises in accordance with expression (2.140). Picture on the phase plane is analogous to Fig. 1.14, but ellipse is extracted in the opposite direction, since x grows/rises with $dx/d\tau>0$. The geometric calculation, analogous to the calculation, given in § 1.4, with the use of formulas (2.138), (2.139), leads to the expression

$$\xi_1 = 1 + a\sqrt{1+\frac{1}{4}a^2} + \frac{1}{2}a^2, \qquad (2.261)$$

where

$$a = \frac{\nu}{\cos\varepsilon}\cdot\frac{l}{S} \qquad (2.262)$$

(ν, ε - the parameters of channel at the point of the rupture of regularity). It is possible to show that value ξ_1 proves to be minimum, when channel is brought at the point where $\varepsilon=0$. Actually/really, ξ_1 - monotonically increasing function a, so that $\xi_1=\xi_{1\text{мин}}$ when $a=a_{\text{мин}}$. From formulas (2.154), (2.262) we have

$$a = |T_{21}|\frac{l}{S\sin\mu}\,. \qquad (2.262a)$$

Parameter a attains the minimum with

$$\frac{d|T_{21}|}{d\tau}=0;\quad \frac{d^2|T_{21}|}{d\tau^2}>0.$$

Let us examine for simplicity channel FD. τ - phase of period, calculated off the beginning of the focusing section. By the multiplication of the conformable matrixes

$$T(\tau) = P_\tau \bar{P} P_{1/2-\tau}$$

we will obtain

$$T_{21} = -2K\,[\text{ch}\,K\sin K - \text{sh}\,K\cos(4K\tau - K)];$$
$$\bar{T}_{21} = -2K\,[\text{ch}\,(4K\tau-K)\sin K - \text{sh}\,K\cos K].$$

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Hence

$$\frac{d\,|T_{21}|}{d\tau} = 8K^2 \operatorname{sh} K \sin(4\tau - 1)K;$$

$$\frac{d\,|\bar{T}_{21}|}{d\tau} = 8K^2 \operatorname{sh}(4\tau - 1)K \sin K.$$

Both functions $|T_{21}(\tau)|$ and $|\bar{T}_{21}(\tau)|$ have an extremum at point $\tau = 1/4$, the extremum corresponding to the minimum

$$\frac{d^2\,|T_{21}|}{d\tau^2} = \frac{d^2\,|\bar{T}_{21}|}{d\tau^2} = 32K^3 \operatorname{sh} K > 0.$$

However, a similar result is obtained also for the periods of other types.

Idle gaps/intervals lead to the essential disagreement/mismatch of beam. Thus, in the linear accelerator of protons I-100 the idle gap/interval between the resonators falls to the points of the extrema of matched envelope. In this accelerator $\frac{S}{\beta\lambda} = 2$; $\nu_3 = 1{,}9$. Hence with $1/\beta\lambda = 1$ we have $\xi_1 = 2{,}5$; $r_{макс} = 1{,}58\, r_{согл}$. With $1/\beta\lambda = 0.5$ we have $\xi_1 = 1{,}6$; $r_{макс} = 1{,}27\, r_{согл}$.

b. Displacement of group of lenses relative to axis. The simultaneous displacement of the group of lenses can be connected, for example, with the displacement of the foundation beam/gully to which is fastened the part of the lenses of the focusing channel. Let us assume the amount of displacement is equal to Δx, the displacement occurring at point $\varepsilon = 0$. Let us assume that is displaced the section of the channel, on which is placed more than one transverse vibration. This shift/shear corresponds to the instantaneous displacement of center of oscillation. Picture on the phase plane is analogous to Fig. 1.13. The amplitude of oscillations grows/rises by value

$$\Delta A = 2\Delta x,$$

but the diameter of beam - by 4 Δx. Displacement tolerance of the group of lenses usually is established/installed within the limits of 0.1-0.3 mm.

c. Displacement of one lens. The considerable displacement of one lens can occur, for example, when terminal span half-tube is fastened to the end-type wall of resonator and its adjustment is respectively hindered/hampered. With the transverse displacement of lens to value Δx the beam falls into the supplementary magnetic field B~~GΔx, which deflects/diverts particles from the initial direction. Parasitic slope deviation of beam, caused by the displacement of lens, is determined for the thin lens by expression (2.209):

$$\Delta \frac{dx}{d\tau} = \frac{S}{D} K^2 \Delta x.$$

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This increase in the inclination/slope is general/common/total for all particles and leads to the fact that the beam begins to oscillate as whole (Fig. 2.22). In particles appears coherent component of oscillations. Since $M=\frac{1}{v}X$, the appearing in this case increase in the amplitude is equal

$$\Delta A=\frac{1}{v_{\text{мин}}}\Delta\frac{dx}{d\tau}. \quad (2.263)$$

It is assumed that at the end of displaced lens $\varepsilon=0$, which corresponds to the worse case.

d. random errors in focusing system. Let us assume that the random errors in the arrangement of quadrupole lenses and in the gradients of focusing fields are distributed in all periods according to one and the same normal law with the mathematical expectation, equal to zero. In this case the mathematical expectation of an increase in the amplitude of transverse vibrations is also equal to zero. The evaluation of the effect of random errors we will produce according to the standard deviation of an increment in the amplitude from zero. Entire calculation let us conduct for the particles with certain average amplitude of oscillations, but not for envelope of particles, since if we conduct calculation for the envelope, then after each random error they would be oriented to the worst particle and would be obtained the too close tolerances, not justified virtually. Allowances depend substantially on the structure of the focusing period. Therefore everywhere, where it is necessary, let us point out, for what structure is obtained the specific allowance. All errors let us relate to one phase of period. For this phase let us accept the middle of the focusing section in which $v=v_{\text{мин}}$.

We differentiate the equation of trajectory (2.98). Taking into account expression (2.155), we obtain

$$x(\tau) = A(\tau)\cos[\Phi(\tau) + \Theta];$$
$$\frac{dx}{d\tau}(\tau) = -\nu(\tau) A(\tau)\sin[\Phi(\tau) + \varepsilon(\tau) + \Theta]. \tag{2.264}$$

The modulus/module of Floquet's function is connected with the symbol of the amplitude of oscillations A(τ). From equations (2.264) it follows:

$$A^2 = \frac{1}{\cos^2\varepsilon}\left[x^2 + \frac{1}{\nu^2}\left(\frac{dx}{d\tau}\right)^2 + 2\frac{\sin\varepsilon}{\nu}x\frac{dx}{d\tau}\right]. \tag{2.265}$$

let the particle trajectory in the assigned phase of period test/experience disturbance/perturbation δx and $\delta(dx/d\tau)$. The amplitude of oscillations will increase by value δA.

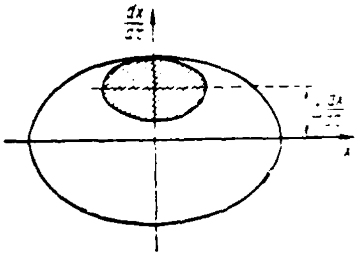

Fig. 2.22.

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We differentiate expression (2.265) and it is averaged on all particles, considering all phases of transverse vibrations equally probable to

$$\overline{(\delta A)^2} = \frac{1}{2\cos^2\varepsilon}\left[\overline{(\delta x)^2} - \frac{1}{v^2}\overline{(\delta \dot{x})^2} + 2\frac{\sin\varepsilon}{v}\overline{\delta x\,\delta \dot{x}}\right].$$

If we carry errors to the middles of the focusing sections where $\varepsilon = 0,\ v = v_\Phi$, then

$$\overline{(\delta A)^2} = \frac{1}{2}\left[\overline{(\delta x)^2} + \frac{1}{v_\Phi^2}\overline{(\delta \dot{x})^2}\right]. \qquad (2.266)$$

At the medium frequency fixed/recorded along the axis of channel of the disturbance/perturbation of the amplitude transverse vibrations do not attenuate. Afterward N_Φ the periods of focusing field the mean square of the disturbance/perturbation of amplitude is equal to

$$\overline{(\Delta A)^2} = \frac{N_\Phi}{2}\left[\overline{\delta x^2} + \frac{1}{v_\Phi^2}\overline{\delta \dot{x}^2}\right], \qquad (2.267)$$

if disturbances/perturbations in the different periods are considered independent variables and equally probable. Let us sum up disturbances/perturbations on the basis of all independent sources of errors in each period

$$\delta x = \Sigma \Delta x; \quad \delta \dot{x} = \Sigma \Delta \dot{x}.$$

As a result we will obtain

$$\Delta A = \sqrt{\frac{\sqrt{\beta}}{2}\left[\Sigma\ \Delta x^2 - \frac{1}{\sqrt{\beta}}\Sigma\ \Delta \dot{x}\right]}. \qquad (2.268)$$

Let us connect now values Δx and Δ(dx/dτ) with errors in the electrical and design parameters in each period. Let us consider the following basic sources of errors: 1) the parallel displacement of the magnetic axis of lens relative to the axis of channel; 2) the inclination/slope of the magnetic axis of lens relative to the axis of channel; 3) the rotation of the median axes of lens around the longitudinal axis of channel; 4) the divergence of the gradient of focusing field from the nominal value. For simplification in the problem we will not examine the longitudinal displacement of lenses relative to calculated position. The effect of latter/last error is small. For example, in the accelerating system with drift tubes the displacement tolerance of tubes, connected with the longitudinal vibrations of particles, proves to be several orders harder than displacement tolerance of quadrupole lenses. Let us introduce the perturbing factors into the equation of motion (2.62). First, there are disturbances/perturbations, which displace beam as whole. These disturbances/perturbations are connected with the departure/attendance of zero-field from the optical axis of channel.

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If coordinate x_1 is counted off from the magnetic axis of lens, then

$$x_1 = x - \varkappa(\tau).$$

where $\varkappa(\tau)$ characterizes the divergence of magnetic axis from the optical. In the second place, there are parametric disturbances/perturbations, which change function $Q(\tau)$:

$$Q_1(\tau) = Q(\tau) + \delta Q(\tau).$$

Thus, the equation of the disturbed motion takes the form

$$\frac{d^2x}{d\tau^2} + Q(1-\alpha)(x-\varkappa) = 0, \qquad (2.269)$$

where

$$\alpha = \frac{\delta Q}{Q}. \qquad (2.270)$$

Taking into account the smallness of values $\varkappa$, α, equation (2.269) can be simplified:

$$\frac{d^2x}{d\tau^2} + Q(1+\alpha)x = Q\varkappa. \qquad (2.269a)$$

Parametric effects are connected with the rotation of median axes and the errors in the gradient. Parametric disturbances/perturbations act on the particles differently, during the dependence on the amplitude and the phases of the fluctuations of each particle, and their resultant action is evinced by an increase in the transverse sizes/dimensions of beam. The divergences of magnetic axis from the optical cause the oscillations of beam as whole. We will search for the solution of equation (2.269a) and in the form

$$x(\tau) = x_0(\tau) + \Delta x(\tau).$$

where function $x_0(\tau)$ - particle trajectory in the ideal channel. Initial conditions for $x(\tau)$ and $x_0(\tau)$ at the entrance of each lens coincide:

$$\Delta x(0)=0; \quad \Delta \frac{dx}{d\tau}(0)=0.$$

Let us assume that occurs the parallel displacement of the magnetic axis of lens relative to the axis of channel to value Δr_0. Equation (2.269a) is reduced in this case to the form

$$\frac{d^2x}{d\tau^2}+Qx=Q\Delta r_0.$$

We approximate focusing fields by "square wave". Then toward the end of focusing lens (τ=D/S)

$$\begin{aligned}&\Delta x_\Phi=(1-\cos K)\,\Delta r_0;\\ &\Delta \frac{dx_\Phi}{d\tau}=\frac{S}{D}K\sin K\Delta r_0.\end{aligned} \qquad (2.271)$$

Solutions (2.271) are obtained analogously with expression (2.164). For the defocusing lens are valid the same formulas with replacement of K by expression iK.

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Subsequently let us rate/estimate allowances in the approximation/approach of thin lenses $K^2 \ll 1$. Because of a good convergence of the obtained series/rows the approximation/approach of thin lenses in this case it is possible to use at any values of $K^2<1$. In this approximation/approach

$$\begin{aligned}&\Delta x_\Phi\approx\frac{1}{2}K^2\Delta r_0; \quad \Delta\frac{dx_\Phi}{d\tau}\approx\frac{SK^2}{D}\Delta r_0;\\ &\Delta x_Д\approx-\frac{1}{2}K^2\Delta r_0; \quad \Delta\frac{dx_Д}{d\tau}\approx-\frac{SK^2}{D}\Delta r_0.\end{aligned} \qquad (2.272)$$

If occurs the inclination/slope of the magnetic axis of lens relative to the axis of channel without the supplementary displacement, then function $\varkappa(\tau)$ takes the form

$$\varkappa(\tau) = \left(\frac{2S}{D}\tau - \frac{D}{2S}\right)\Delta r_{\text{к}}. \qquad (2.273)$$

where $\Delta r_{\text{к}}$ - absolute shift of each end/lead of magnetic axis. Integration of equation (2.269a) in the section of the focusing lens gives in the approximation/approach of the thin lenses

$$\Delta x_{\Phi} = -\frac{1}{6}K^2\Delta r_{\text{к}};\quad \Delta\frac{dx_{\Phi}}{d\tau} = \frac{SK^4}{12D}\Delta r_{\text{к}};$$
$$\Delta x_{\text{Д}} = \frac{1}{6}K^2\Delta r_{\text{к}};\quad \Delta\frac{dx_{\text{Д}}}{d\tau} = \frac{SK^4}{12D}\Delta r_{\text{к}}. \qquad (2.274)$$

Let us assume two lenses of different signs are structurally/constructurally united. Then the error of their general/common/total inclination/slope is accompanied by the errors of shift. In this case

$$\Delta x_{\Phi} = \frac{1}{2}K^2\frac{\Delta r_{\text{к}}}{2} - \frac{1}{6}K^2\frac{\Delta r_{\text{к}}}{2} = \frac{1}{6}K^2\Delta r_{\text{к}};$$
$$\Delta\frac{dx_{\Phi}}{d\tau} = \frac{SK^2}{D}\cdot\frac{\Delta r_{\text{к}}}{2} + \frac{SK^4}{12D}\cdot\frac{\Delta r_{\text{к}}}{2} \approx \frac{SK^2}{2D}\Delta r_{\text{к}}.$$

Let us examine now parametric disturbances/perturbations. Deducting the equation of undisturbed motion (2.62) from equation (2.269a), we obtain for $\Delta x(\tau)$ the equation

$$\frac{d^2}{d\tau^2}(\Delta x) + Q\Delta x = -\alpha Q x_0(\tau). \qquad (2.275)$$

Solution we will search for in the approximation/approach of thin lenses. In this case the right side of equation (2.275) can be considered approximately constant and assumed x_0=A, where A -

average/mean value/significance of envelope in the period of focusing field. The solution of equation (2.275) toward the end of the focusing lens takes the form

$$\Delta x_\Phi = -\frac{\alpha}{2} K^2 A; \quad \Delta \frac{dx_\Phi}{d\tau} = -\frac{\alpha S}{D} K^2 A. \qquad (2.276)$$

For the defocusing lens K^2 is reversed the sign.

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If there is an error in the field gradient, then according to expressions (2.163), (2.270),

$$\alpha = \frac{\delta G}{G}. \qquad (2.277)$$

But if the median axes of quadrupole lens around the axis of channel sw..g through angle $\Delta\psi$, then the coordinate of certain point relative to median axes ξ, η they will be connected with the coordinates relative to the axes of channel with the relationship/ratio

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos\Delta\psi & \sin\Delta\psi \\ -\sin\Delta\psi & \cos\Delta\psi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Assuming that the lens is ideal, it is possible to write the potential of the field of the turned lens in the form

$$U_\Phi = G\xi\eta,$$

or in the old coordinates

$$U_\Phi = Gxy\cos 2\Delta\psi - \frac{1}{2} G(x^2 - y^2)\sin 2\Delta\psi.$$

Then

$$B_y = Gx\cos 2\Delta\psi + Gy\sin 2\Delta\psi.$$

In plane XOZ

$$B_y \approx Gx[1 - 2(\Delta\psi)^2].$$

Hence we have

$$\alpha = -2(\Delta\psi)^2. \qquad (2.278)$$

Let us convert the disturbances/perturbations of trajectories (2.272), (2.274), (2.276) with the aid of the conformable matrixes to the middle of the focusing section where $\nu = \nu_\Phi$.

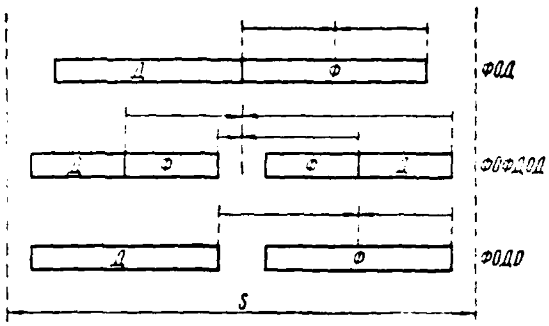

Fig. 2.23.

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Fig. 2.23 shows the diagram of recalculation for different type focusing periods. The converted disturbances/perturbations let us note by the asterisks

$$\Delta x_{\Phi} \to \Delta x^*_{\Phi}; \quad \Delta \dot{x}_{\Phi} \to \Delta \dot{x}^*_{\Phi}$$
$$\Delta x_{Д} \to \Delta x^*_{Д}; \quad \Delta \dot{x}_{Д} \to \Delta \dot{x}^*_{Д}. \qquad (2.279)$$

Let us drop/omit recalculations, after assuming that they are produced. Subsequently let us give the final results of calculation. The contribution of different errors depends substantially on the structure of the period of focusing field. For example, in the system FODO lenses are fastened independently and the random errors in their position are not correlated. In the system FOD the lenses are structurally/constructurally united, so that positional errors store/add up. System FOFDOD is the combined case. The correlation between errors in the gradient depends on the diagram of supply. If the pair of lenses is supplied by current consecutively/serially, then this corresponds to system FOD. Complete root-mean-square disturbances/perturbations in the period of the type FODO (errors are independent)

$$\langle \Delta x^* \rangle = \sqrt{\langle \Delta x^*_{\text{Ф}} \rangle^2 + \langle \Delta x^*_{\text{Д}} \rangle^2},$$

$$\langle \Delta \dot{x}^* \rangle = \sqrt{\langle \Delta \dot{x}^*_{\text{Ф}} \rangle^2 + \langle \Delta \dot{x}^*_{\text{Д}} \rangle^2}.$$

Since the formulas for Δx^* and $\Delta \dot{x}^*$ are analogous, let us extract formulas only for Δx^*. For the period of the type FOD we have

$$\langle \Delta x^* \rangle = \langle \Delta x^*_{\text{Ф}} - \Delta x^*_{\text{Д}} \rangle.$$

For the period of the type FOFDOD (see Fig. 2.23)

$$\langle \Delta x^* \rangle = \sqrt{\langle \Delta x^*_{\text{Ф}} + \Delta x^*_{\text{Д}} \rangle^2_{\text{л}} + \langle \Delta x^*_{\text{Ф}} - \Delta x^*_{\text{Д}} \rangle^2_{n}}.$$

Index l relates to the left pair of lenses; index p - to the right.

Calculations employing the given above procedure give the following results. In the systems FOD and FOFDOD the errors of lateral misalignment and parametric disturbances/perturbations in the first approximation, average out, since $\Delta x_{\text{Ф}}$ and $\Delta x_{\text{Д}}$ have different signs. The slant errors in these systems store/add up, since with the change by the places for that focusing and defocusing of lenses reverse the signs both in the value K^2 and in the average/mean shift of the axis of each lens. The effect strictly of inclination/slope is unessential, but the axes obtained with the inclination/slope of the pair of the lenses of shift give considerable contribution, moreover the errors of shift in this case are not compensated, but they store/add up.

Thus, in the systems FOD and FOFDOD of the disturbance/perturbation of trajectories, connected with the errors of parallel shift and the parametric errors, it is less than in the system FODO. The disturbances/perturbations, connected with the inclination/slope of axes, in the systems FOD and FOFDOD are more than in the system FODO.

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The thermal deformations of the rods of drift tubes usually lead to lateral misalignment. In such cases the best results give systems FOD and FOFDOD. The advantage of system FOFDOD over the system FOD consists in the fact that the period FOFDOD is symmetrical; channel FOFDOD it is easy to begin so that the matched initial conditions would correspond to the crossover of bundle. During a good temperature stabilization of rods the system FODO is more preferable, since it is structurally/constructurally simpler.

Let us give the summary of the formulas, obtained as a result of calculation employing the procedure indicated. Rms value of an increment in the amplitude of transverse vibrations

$$\langle \Delta A \rangle = \sqrt{\frac{N_\Phi}{2}\left[\Sigma \langle \Delta x^* \rangle^2 + \frac{1}{\nu_\Phi^2} \Sigma \langle \Delta \dot{x}^* \rangle^2\right]}.$$

Components of disturbances/perturbations the following:

1) the inclination/slope of the longitudinal axis of the lens

$$\langle\Delta x^*\rangle = a_1K^2\langle\Delta r_\kappa\rangle;\quad \langle\Delta\dot{x}^*\rangle = b_1K^2\langle\Delta r_\kappa\rangle;$$

2) lateral misalignment of the lens

$$\langle\Delta x^*\rangle = a_2K^2\langle\Delta r_0\rangle;\quad \langle\Delta\dot{x}^*\rangle = b_2K^2\langle\Delta r_0\rangle;$$

3) the rotation of the median axes of the lens

$$\langle\Delta x\rangle^* = 4a_2K^2A\sqrt{\overline{(\Delta\psi)^2}};\quad \langle\Delta\dot{x}^*\rangle = 4b_2K^2A\sqrt{\overline{(\Delta\psi)^2}};$$

4) the divergence of gradient from the nominal value

$$\langle\Delta x^*\rangle = a_2K^2A\left\langle\frac{\delta G}{G}\right\rangle;\quad \langle\Delta\dot{x}^*\rangle = b_2K^2A\left\langle\frac{\delta G}{G}\right\rangle.$$

The values/significances of coefficients of a_1, b_1, a_2, b_2:

1) system FOD

$$a_1 = \sqrt{2}\left(1-\frac{K^2}{4}\right)^{1/2};\quad b_1 = \left(1-\frac{K^2}{4}\right)^{1/2};$$

$$a_2 = \left(1-\frac{K^2}{12}\right)^{1/2};\quad b_2 = \frac{K^2}{2};$$

2) system FOFDOD (g - gap length between the lenses; see Fig. 2.23)

$$a_1 = \sqrt{2}\left[\left(1+\frac{g}{2D}\right)^2+\frac{K^2}{10}\left(1-5\frac{g}{D}-\frac{5}{2}\frac{g^2}{D^2}\right)\right]^{1/2};$$

$$b_1 = \sqrt{2}\left[1-\frac{K^2}{4}\cdot\frac{g}{D}\right]^{1/2};$$

$$a_2 = \sqrt{2}\left[1-\frac{3}{4}K^2\frac{g}{D}\right]^{1/2};$$

$$b_2 = K^2\left[1+\left(1-\frac{g}{2D}\right)^2+\frac{K^2}{4}\left(1-\frac{8}{3}\frac{g}{D}\right)\right]^{1/2};$$

3) system FODO

$$a_1 = \frac{1}{3\sqrt{2}}\left[1 + \frac{K^2}{4}\left(1 + 2\frac{g}{D}\right)\right]^{1/2};$$

$$b_1 = \frac{K^4}{\sqrt{2}}10^{-2}\left[1 + \left(1 + 6\frac{g}{D}\right)^2\right]^{1/2};$$

$$a_2 = \left[\left(1 + \frac{g}{D}\right)^2 - \frac{K^2}{6}\left(1 + \frac{5}{2}\frac{g}{D} + \frac{3}{2}\frac{g^2}{D^2}\right)\right]^{1/2};$$

$$b_2 = \sqrt{2}\left[1 - \frac{K^2}{4}\left(1 + 2\frac{g}{D}\right)\right]^{1/2}.$$

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Due to low values ν_Φ the contribution of sum $\langle \Delta x^* \rangle$ usually is substantially less than the contribution, introduced into an increment in the amplitude by sum $\langle \Delta \dot{x}^* \rangle$:

$$\langle \Delta A \rangle \approx \frac{1}{\nu_\Phi}\sqrt{\frac{N_\Phi}{2}\Sigma \langle \Delta \dot{x}^{*2} \rangle}. \qquad (2.280)$$

Therefore the disturbances/perturbations of trajectories, connected with the random errors, The less, the higher ν_Φ and the less the hardness of lenses. Hence it is apparent that working value cos μ should be selected near maximum ν_Φ, decreasing, how this is possible without noticeable reduction ν_Φ, the value of hardness. Above they arrived at the same conclusion, on the basis of the considerations about the channel capacity and power scattered in the lenses.

From formula (2.280) it follows that the allowances for the random errors in essence depend on coefficients of b_1, b_2. Table 2.2 gives the values of coefficients of b_1, b_2 for different structures in the zero approximation in K^2. Table 2.2 shows the effect of different random errors in the focusing channel with the determinate structures of period.

In practice during the calculation of allowances should be utilized terms not lower than the first approximation on K^2. Let us give for the orientation the summary of allowances for the strong-focusing system of the proton linear accelerator I-100 with the exit energy 100 MeV [14]. In this accelerator is used the system FODO (see Table 2.1) with the series feed of the lenses of one period. The length of period $S=2\beta\lambda$; medium frequency corresponds $\cos\mu_s \approx 0.5$; the average coefficient of clearance (1.23) α~~0.25.

Table 2.2.

(1) Коэффициенты	ФОД	ФОФДОД	ФОДО
b_1	1	1.4	0
b_2	0	0	1.4

Key: (1). Coefficients.

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Allowances are distributed as follows: $\Delta r_0 = \Delta r_к = 120$ μ; $\Delta\psi = 1°$; $\left(\frac{\delta G}{G}\right) = 1\%$. Allowances for the divergence of the magnetic axis of lens from the axis of channel Δr_0, $\Delta r_к$ are divided into two parts: allowance for the position of the magnetic axis of lens relative to the centerline of drift tube 90 μ; allowance for the position of the centerline of drift tube relative to the axis of channel - 50 μ. Expected increase in the amplitude of transverse vibrations of particles at the length of the accelerator

$$\langle\Delta A\rangle = 0{,}83\sqrt{N_Ф}\ мм.$$

e. Nonlinearity of focusing field. In the real cases it is impossible to achieve absolutely precise beam matching with the focusing channel. Let us assume in the channel is spread the unmatched beam with the ratio of effective phase volume to the true

$$\xi_1 = \frac{V_{эфф}}{V_п}. \qquad (2.281)$$

If channel is comprised of the ideal quadrupole lenses, then the phase volume of beam will rotate within the effective volume without the essential deformations. Besides rotation it can take place and another form of the motion of true phase volume within the effective - wandering due to the coherent divergences, which appear, in particular, due to the shift of the magnetic axes of lenses from the axis of channel. In the linear focusing fields all these effects do not lead to the irreparable deformations of phase volume, with the virtually equivalent to its increase. But if focusing fields are nonlinear, then phase volume is distorted, since the frequencies of transverse vibrations of particles with the different amplitudes are different. As an example Fig. 2.24 shows the form of the phase volume of beam after its passage through the accelerator with a number of periods of focusing field $N_\Phi = 80$ and $\cos\mu_n = 0.3$, where μ_n - frequency of small (linear) transverse vibrations. The ratio of effective phase volume to the true is accepted by equal to $\xi_1=2$, and the divergence of field from the linear value/significance on the radius, equal to the amplitude of oscillations, composes 1.1o/o. As can be seen from Fig. 2.24, the nonlinearity of the fields of lenses leads to the distortion of phase volume. Bulk of particles continues to remain within the limits of the ellipse, which limits phase volume at the entrance of accelerator. But the particles, which caught into the ejections (not shaded in Fig. 2.24), should be considered lost, since after the output of beam from the channel they, as a rule, no longer can be used.

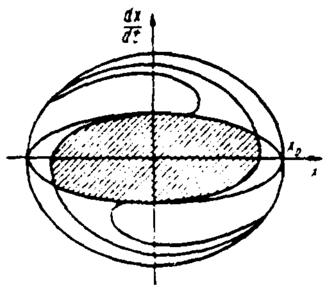

Fig. 2.24.

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Let us calculate the portion of the lost particles, on the basis of the fact that the phase density is distributed evenly. Let us disregard/neglect for simplicity the defocusing action of accelerating gaps and will examine transverse vibrations of particles in plane XOZ of the channel, comprised of the imperfect lenses. Substituting field expression of imperfect lens (2.32) in the first equation of motion (2.51) and converting/transferring to the dimensionless variable τ, we will obtain

$$\frac{d^2x}{d\tau^2}+Q(\tau)\left[x+\sum_{m=1}\frac{A_{m0}}{G}x^{4m+1}\right]=0, \qquad (2.282)$$

where G - value of field gradient on the axis of lens. In the first approximation, the ratio of percentage distortions A_{m0} (determined in essence by the profile/airfoil of pole) to the value of gradient on the axis from the longitudinal coordinate does not depend. We will search for the solution of equation (2.282) in smooth approximation/approach (2.222), after being restricted to the first two nonlinear terms. By assuming/setting

$$x^5 \approx X^5(1-5q);$$
$$x^9 \approx X^9(1-9q)$$

and by averaging equation (2.282) for the period of focusing field, let us arrive at the following equation for the slow component of the trajectory:

$$\frac{d^2X}{d\tau^2} - \mu_\pi^2\left(X - 5\frac{A_{10}}{G}X^5 - 9\frac{A_{20}}{G}X^9\right) = 0. \qquad (2.283)$$

The medium frequency of small oscillations μ_π is as before determined by equality (2.229), and function $q(\tau)$ - by equation (2.224) and by conditions (2.226), (2.227). Let us multiply equation (2.283) by $dx/d\tau$ and will integrate. As a result we will obtain the first integral

$$\left(\frac{dX}{d\tau}\right)^2 - \mu_\pi^2 V(X) = 0. \qquad (2.284)$$

where

$$V(X) = X_0^2 - X^2 + \frac{5}{3}\frac{A_{10}}{G}(X_0^6 - X^6) - \frac{9}{5}\cdot\frac{A_{20}}{G}(X_0^{10} - X^{10}). \qquad (2.285)$$

Here X_0 - amplitude of the oscillations of the slow component of trajectory. According to expression (2.284),

$$d\tau = \frac{1}{\mu_\pi}\cdot\frac{dX}{\sqrt{V(X)}}.$$

Hence we obtain the period of oscillations of the slow component of trajectory with an amplitude of X_0

$$\frac{2\pi}{\mu} = \frac{4}{\mu_\pi}\int_0^{X_0}\frac{dX}{\sqrt{V(X)}}. \qquad (2.286)$$

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Let us substitute function (2.285) into integral (2.286) and is decomposed denominator in the series/row according to the degrees of the low values A_{10}/G and A_{20}/G. Being limited to the linear terms of resolution and by integrating, we will obtain

$$\frac{\mu-\mu_{\pi}}{\mu_{\pi}}=1.6\frac{A_{10}}{G}X_0^4-2.2\frac{A_{20}}{G}X_0^8. \qquad (2.287)$$

Let us note that values

$$\Delta B_5=A_{10}X_0^5; \qquad \Delta B_9=A_{20}X_0^9$$

are, according to expression (2.32), the divergences of field on radius X_0, by the caused corresponding harmonics, from the ideal linear value/significance of $B=GX_0$. Thus,

$$\frac{\mu-\mu_{\pi}}{\mu_{\pi}}=1.6\frac{\Delta B_5}{B}+2,2\frac{\Delta B_9}{B}\approx 2\frac{\Delta B}{B}. \qquad (2.288)$$

The relative deflection of frequency from the frequency of small is approximately equal to the doubled relative deflection of field from the linear value/significance on the radius, equal to the amplitude of transverse vibrations.

Fig. 2.25 shows the dependences of a number of lost particles on value $\Delta B/B$ at the different values ξ_1 for the accelerator whose

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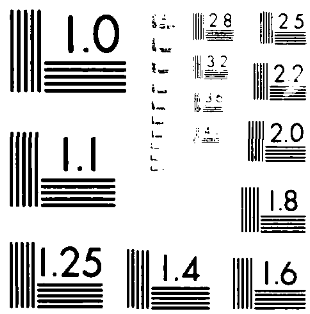

MICROCOPY RESOLUTION TEST CHART

parameters are stipulated above. Curves are obtained by the calculation of areas on the graphs/curves, analogous to Fig. 2.24 [52]. From the graphs on Fig 2.25 it is evident that a number of lost particles noncritically depends on the value of disagreement/mismatch ξ_1. If we allow loss by 10o/o of particles and value of disagreement/mismatch not worse ξ_1=2, then allowance for the divergence of field will compose ΔB/B~~0.5o/o. Taking into account the linear dependence of phase change of small oscillations on τ at the length of accelerator, it is possible to obtain formula for evaluating the standard deviation of field from the linear within the limits of the assigned useful aperture

$$\frac{\Delta B}{B} \approx \frac{45}{N_\Phi}\%, \qquad (2.289)$$

where N_Φ - total number of periods of focusing field at the length of accelerator.

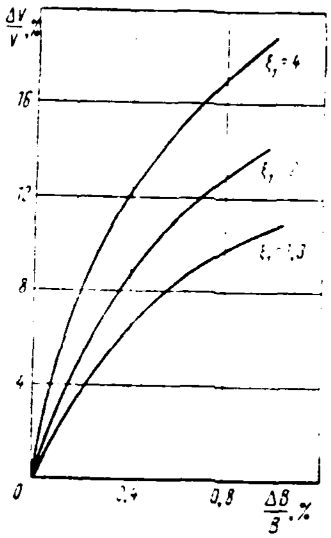

Fig. 2.25.

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S2.9. Focusing by longitudinal magnetic field.

The basic advantages of the system of focusing, which uses a longitudinal magnetic field of solenoids, in comparison with the strong focusing are structural/design simplicity and facilitation of requirements for the allowances. At the same time focusing by longitudinal magnetic field possesses the essential deficiencies/lacks which are examined subsequently. Due to these deficiencies/lacks the longitudinal focusing fields in the contemporary linear accelerators find thus far only limited application.

During calculations of particle dynamics we will consider longitudinal magnetic field uniform. Real accelerating field let us replace with the equivalent traveling wave. It is obvious that the obtained results will relate to the particle acceleration in the traveling wave and to the particle acceleration in the field of standing waves with the smallness of a partial increase in energy.

The components of electric field take the form [see expressions (1.43), (2.9)]

$$E_z = E\cos\omega\left(t - \int_0^z \frac{dz}{v_s}\right);$$

$$E_x = -\frac{x}{2}\cdot\frac{\partial E_z}{\partial z}; \qquad E_y = -\frac{y}{2}\cdot\frac{\partial E_z}{\partial z}.$$

Components of the magnetic field

$$B_x = -\frac{y}{2c^2}\cdot\frac{\partial E_z}{\partial t}; \quad B_y = \frac{x}{2c^2}\cdot\frac{\partial E_z}{\partial t}; \quad B_z = B.$$

Here B - external focusing field. Field. By hypothesis B=const. In traveling wave $\frac{\partial E_z}{\partial t} = -v_s\frac{\partial E_z}{\partial z}$. Projecting/designing equation of motion (2.5) to the transverse coordinate axes and substituting expressions for the components of electrical and magnetic fields, we obtain

$$\begin{aligned} \frac{dp_x}{dt} &= eB\frac{dy}{dt} - \frac{ex}{2}(1-\beta^2)\frac{\partial E_z}{\partial z}; \\ \frac{dp_y}{dt} &= -eB\frac{dx}{dt} - \frac{ey}{2}(1-\beta^2)\frac{\partial E_z}{\partial z}. \end{aligned} \qquad (2.290)$$

The equation of longitudinal vibrations retains form (2.50), if we disregard/neglect the small terms x(dx/dt) and y(dy/dt). Assuming that the mass of particles is changed sufficiently slowly, let us rewrite equations (2.290) in the form

$$\frac{d^2x}{dt^2} = \frac{eB}{m_0\gamma}\cdot\frac{dy}{dt} - \frac{e}{2m_0\gamma^3}\cdot\frac{\partial E_z}{\partial z}x;$$
$$\frac{d^2y}{dt^2} = -\frac{eB}{m_0\gamma}\cdot\frac{dx}{dt} - \frac{e}{2m_0\gamma^3}\cdot\frac{\partial E_z}{\partial z}y. \quad (2.291)$$

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According to expressions (1.70), (2.12),

$$-\frac{e}{m_0\gamma^3}\cdot\frac{\partial E_z}{\partial z} = \Omega^2\frac{\sin\varphi}{\sin\varphi_s}, \quad (2.292)$$

where Ω - relativistic frequency of small longitudinal oscillations. Let us introduce for the decrease of recordings the designation

$$\Omega_\varphi^2 = \Omega^2\frac{\sin\varphi}{\sin\varphi_s}. \quad (2.293)$$

Value

$$\omega_L = \frac{eB}{2m_0\gamma} \quad (2.294)$$

is frequency of the Larmor precession of particles in the magnetic field. Taking into account designations (2.293), (2.294) we have

$$\frac{d^2x}{dt^2} = 2\omega_L\frac{dy}{dt} + \frac{1}{2}\Omega_\varphi^2 x;$$
$$\frac{d^2y}{dt^2} = -2\omega_L\frac{dx}{dt} + \frac{1}{2}\Omega_\varphi^2 y. \quad (2.295)$$

Together with the equations of motion in Cartesian coordinates (2.295) let us examine equations in polar coordinates r, ψ:

$$x = r\cos\psi; \qquad y = r\sin\psi.$$

Equations of motion in the polar coordinates can be obtained directly from expressions (2.295) by the appropriate replacement of the variable/alternating

$$\frac{d^2r}{dt^2}=r\left(\frac{d\psi}{dt}\right)^2+2\omega_L r\frac{d\psi}{dt}+\frac{1}{2}\Omega_\psi^2 r;$$
$$r\frac{d^2\psi}{dt^2}=-2\omega_L\frac{dr}{dt}-2\frac{dr}{dt}\cdot\frac{d\psi}{dt}. \quad (2.296)$$

Let us multiply the second equation by r. Assuming/setting $\omega_L=\text{const}$, we obtain

$$\frac{d}{dt}\left(r^2\frac{d\psi}{dt}+\omega_L r^2\right)=0.$$

Thus, one of the first integ~als of equations (2.296) is equal to

$$r^2\left(\frac{d\psi}{dt}+\omega_L\right)=M, \quad (2.297)$$

where M=const. Ravenstvo (2.297) is called Busch's theorem. It is possible to show that Busch's theorem (2.297) is valid in the general case when ω_L - variable quantity [64]. Let there be at moment/torque t-0: $r=r_0$; $\frac{d\psi}{dt}=\dot{\psi}_0$; $\omega_L=\omega_L^0$. Then

$$M=r_0^2(\dot{\psi}_0+\omega_L^0).$$

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Let us assume that the particles enter into the focusing channel from the region where there is no magnetic field. In this case the constant value

$$M=r_0^2\dot{\psi}_0$$

makes physical sense of the moment of momentum of particle with the single mass relative to the assigned axis of channel. If particle enters into focusing field, without having initial rotation, then in the field it will rotate with an angular velocity of

$$\frac{d\psi}{dt} = -\omega_L. \qquad (2.298)$$

In the homocentric ray, i.e., in the beam, which possesses zero phase volume, the moment of momentum is equal to zero for all particles. In the longitudinal magnetic field all particles of homocentric ray will rotate with identical angular velocity (2.298). When the disordered scatter of transverse thermal velocities is present, there is a scatter along the moment of momentum. Let us assume that the positive and negative values of the moment of momentum are equally probable. Then the average/mean value/significance of the moment of momentum is equal to zero and

$$\overline{\frac{d\psi}{dt}} = -\omega_L.$$

The angular velocities of single particles in the focusing field depend on the instantaneous value of the radius:

$$\frac{d\psi}{dt} = -\omega_L - \frac{M}{r^2}.$$

Let us substitute this expression in first equation (2.296):

$$\frac{d^2r}{dt^2} + \left(\omega_L^2 - \frac{1}{2}\Omega_\varphi^2\right) r - \frac{M^2}{r^3} = 0. \qquad (2.299)$$

Expression (2.299) takes the same form as equation for envelope of particles, obtained in theory of strong focusing. However, the physical sense of equation (2.299) is another - this is equation not for envelope of particles, but for a radius - the vector of any particle relative to the chosen axis. If we place the moment of momentum equal to zero, then equation (2.299) is simplified:

$$\frac{d^2r}{dt^2} + \left(\omega_L^2 - \frac{1}{2}\Omega_\varphi^2\right) r = 0. \qquad (2.300)$$

With the zero initial moment of momentum a radius - the vector of particle varies according to the sinusoidal law with a frequency of

$$\omega_r = \sqrt{\omega_L^2 - \frac{1}{2}\Omega_\varphi^2}. \qquad (2.301)$$

Homocentric ray is confined into the point through each half-period of the radial oscillations

$$\frac{L_r}{2} = \frac{\pi v}{\omega_r}. \qquad (2.302)$$

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In the absence of accelerating field the frequency of radius-vector coincides with the angular frequency.

The solutions of equation (2.299) are the moduli/modules of the corresponding complex solutions of equation (2.300). Consequently, the solutions of equation (2.299) are stable, if are stable the solutions of equation (2.300). Stability condition directly follows from equation (2.300):

$$\omega_L^2 > \frac{1}{2}\Omega^2 \frac{\sin\varphi}{\sin\varphi_s}. \qquad (2.303)$$

Let us require so that the stability of trajectories would be retained in all phases of particles within the limits of the separatrix

$$2\varphi_s < \varphi < -\varphi_s.$$

When $\varphi=-\varphi_s$ condition (2.303) is satisfied automatically. This is connected with the fact that in the positive phases accelerating field focuses particles. When $\varphi=2\varphi_s$ inequality (2.303) is retained, if

$$\omega_L^2>\Omega^2\cos\varphi_s.$$

Since virtually $\sqrt{\cos\varphi_s}\approx 1,$ the latter/last condition can be simplified

$$\omega_L>\Omega. \qquad (2.304)$$

Taking into account that the frequency of small phase oscillations in the process of acceleration and diabatically decreases, beginning from the initial value $\Omega(0)$, it is possible, in particular, to select

$$\omega_L=\Omega(0).$$

In the focusing system with the longitudinal magnetic field analogous with the strong-focusing system occurs the parametric effect of longitudinal vibrations on the radial oscillations. Let us examine the particles, which accomplish small longitudinal vibrations (2.243) and not possessing initial rotation. The radial fluctuations of such particles are described by the equation

$$\frac{d^2r}{dt^2}+\left[\omega_L^2-\frac{1}{2}\Omega^2-\frac{1}{2}\Omega^2\Phi\operatorname{ctg}\varphi_s\sin\Omega t\right]r=0. \qquad (2.305)$$

By the replacement of the independent variable

$$\Omega t=2\pi\tau$$

equation (2.305) is converted to canonical form (2.108), moreover

$$a=\left(\frac{2\omega_L}{\Omega}\right)^2-2;\qquad q=\Phi\operatorname{ctg}\varphi_s.$$

Stability regions are depicted in Fig. 2.11. Assuming/setting, as for the strong-focusing system, $\cos\varphi_s=0{,}8$ and $\Phi=\varphi_s$, we have q=0.86.

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If in the beginning of accelerator is satisfied condition (2.304), which ensures the stability of all particles, then coefficient a proves to be equal to 2 that it guarantees the work higher than first region of parametric resonance. Subsequently the frequency of small decreases. If $\frac{\omega_L}{\Omega}=\text{const},$ then operating point on diagram 2.11 is displaced horizontally to the axis of ordinates. But if $\omega_L=\Omega(0)=\text{const},$ then operating point is displaced upward and to the left, passing, as in the case of strong focusing in $\cos\mu_s=\text{const},$ the region of multiple parametric resonances.

Let us determine a radius of the focused beam depending on the magnetic field strength and phase volume of beam. Equation (2.299) can be rewritten in the more compact form, if we use designation (2.301):

$$\frac{d^2r}{dt^2}+\omega_r^2r-\frac{M^2}{r^3}=0. \qquad (2.299a)$$

The first integral of this equation is the following

$$I=\left(\frac{dr}{dt}\right)^2+\omega_r^2r^2+\frac{M^2}{r^2}. \qquad (2.306)$$

Taking into account expression (2.297) first integral (2.306) can be represented in the form

$$I = v_\perp^2 + \omega_r^2 r^2,$$

where

$$v_\perp = \left(\frac{dr}{dt}\right)^2 + r^2\left[\frac{d\psi}{dt} - \frac{\overline{d\psi}}{dt}\right]^2$$

is the square of the disordered component of complete linear velocity in the plane, perpendicular to the axis of bundle; dr/dt - radial velocity; $r\frac{d\psi}{dt}$ - linear peripheral speed. Let us examine particles with the given value of integral (2.306). The greatest removal/distance from the axis they reach particle with the zero initial moment of momentum with dr/dt=0

$$\omega_r^2 r_{\text{макс}}^2 = I.$$

Let $I = I_{\text{макс}}$ - greatest value/significance of integral (2.306) for the particles in the beam. Then a radius of beam R is determined from the expression

$$R = \frac{\sqrt{I_{\text{макс}}}}{\omega_r}. \qquad (2.307)$$

Let us connect value $I_{\text{макс}}$ with the transverse phase volume of beam. Fig. 2.26 gives phase particle trajectories with given value $I = I_{\text{макс}}$ on plane r, dr/dt. The family of phase trajectories is constructed graphically, on the basis of the relationship/ratio.

$$\frac{dr}{dt} = \sqrt{I - V(r)}.$$

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The function

$$V(r) = \omega_r^2 r^2 + \frac{M^2}{r^2}$$

is the analog of potential energy. The actual values of radial velocity occur with V(r)≤I. With M=0 phase trajectory takes the form of ellipse. With M≠0 phase trajectories fall inside this ellipse. In accordance with that outlined above of maximum removal/distance from the axis they reach particle with M=0. Thus, for solving the question about the sizes/dimensions of beam it suffices to explain the value of the phase volume, occupied by the particles, which do not have initial rotation.

Let us assume that the beam possesses axial symmetry and its phase volume in four-dimensional space x, y, p_x, p_y is limited at the entrance of the focusing channel by the ellipsoid

$$a(x^2+y^2)+b(p_x^2+p_y^2)+ \\ +2c(xp_x+yp_y)=1. \quad (2.308)$$

The projection of phase volume on plane x, p_x exists

$$ax^2+bp_x^2+2cxp_x=1. \quad (2.309)$$

Actually/really, point in the curve, which covers projection, corresp..ds to maximally possible value/significance p_x for each fixed value of x:

$$\frac{\partial p_x}{\partial y}=0; \quad \frac{\partial p_x}{\partial p_y}=0.$$

According to the rule of implicit differentiation

$$\frac{\partial p_x}{\partial y}=-\frac{\partial F}{\partial y}\Big/\frac{\partial F}{\partial p_x}; \quad \frac{\partial p_x}{\partial p_y}=-\frac{\partial F}{\partial p_y}\Big/\frac{\partial F}{\partial p_x},$$

where $F(x, y, p_x, p_y)$ - left side of equation (2.308). The maximum impulse/momentum/pulse p_x occurs at values y, p_y of those determined by system of equations

$$\frac{\partial F}{\partial y}=0; \quad \frac{\partial F}{\partial p_y}=0. \quad (2.310)$$

Producing particular differentiation, we obtain

$$ay+cp_y=0;$$
$$cy+bp_y=0.$$

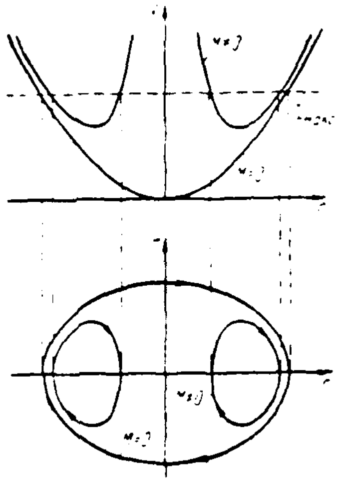

Fig. 2.26.

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Determinant of this system of equations

$$\Delta = ab - c^2.$$

The volume of four-dimensional ellipsoid (2.308) is equal to

$$V_4 = \frac{\pi^2}{2\Delta},$$

so that $\Delta \neq 0$. Hence $y = p_y = 0$ we obtain projection (2.309). Let us pass in equation (2.308) to the polar coordinates

$$ar^2 - b(p_r^2 - p_\psi^2) - 2crp_r = 1;$$

here

$$p_r = m\frac{dr}{dt}; \quad p_\psi = mr\frac{d\psi}{dt}.$$

Particles without the initial rotation occupy on plane r, p_r the region

$$ar^2 - bp_r^2 - 2crp_r = 1. \qquad (2.311)$$

area of which coincides with the projected area (2.309). In accordance with determination (2.2) the invariant phase volume, occupied by such particles, is equal to

$$V_n = \frac{1}{\pi m_0 c} \int p_r dr. \qquad (2.312)$$

If the representative points of particles completely fill region within the maximum phase trajectory, which corresponds $I = I_{\text{макс}}$, M=0 (see Fig. 2.26), then beam is matched with the channel and its radius (2.307) is a constant value. Maximum phase trajectory on plane r, dr/dt is described by the equation of the ellipse

$$\frac{\dot{r}^2}{I_{\text{макс}}} - \frac{r^2}{\frac{I_{\text{макс}}}{\omega_r^2}} = 1 \qquad (2.313)$$

and covers the area

$$\int \dot{r}\, dr = \frac{\pi I_{\text{макс}}}{\omega_r}.$$

Replacing in integral (2.312) radial impulse/momentum/pulse by its expression through the radial velocity, we obtain

$$V_n = \frac{\gamma I_{\text{макс}}}{c\omega_r}. \qquad (2.314)$$

Hence taking into account (2.307)

$$V_n = \frac{\gamma}{c} \omega_r R^2. \qquad (2.315)$$

Channel capacity with a radius of the aperture opening/aperture a is equal to

$$V_K = \frac{\gamma}{c} \omega_r a^2. \qquad (2.316)$$

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The emittance of beam and the acceptance of channel are connected in accordance with the phase volume of beam and channel capacity with relationships/ratios (2.4), (2.127).

To match beam with the longitudinal focusing field it is simpler than with the strong-focusing channel, since in the longitudinal field to match it is necessary only the diameter of beam at the entrance.

If the region, occupied by the representative points of particles on plane r, dr/dt, does not coincide with ellipse (2.313) with some value/significance $I_{макс}$, then beam proves to be mismatched with the channel. Let us describe around the phase volume of unmatched beam on plane r, dr/dt ellipse with the relation of semi-axes ω_r. The area, included by this ellipse, corresponds to effective phase volume $V_{п.эфф}$. True phase volume rotates within the effective, so that the envelope of unmatched beam oscillates with a frequency of $2\omega_r$. If the ratio of the effective volume to the true comprises $\xi_1>1$, then the maximum size of unmatched beam will be $\sqrt{\xi_1}$ times more than a radius of matched beam with the same phase volume.

In the presence of high-frequency accelerating field the permissible value of the induction of the longitudinal focusing field is bounded below by stability condition (2.304). Let us examine two possible versions of the selection of focusing field.

1. Frequency of Larmor precession at each moment of time coincides with instantaneous value of frequency of small longitudinal vibrations

$$\omega_L = \Omega(t). \tag{2.317}$$

With an increase in the energy of particles the required frequency of Larmor precession will descend and respectively it will be possible to decrease the induction of focusing field. In this case, according to expression (2.301), for the synchronous particle

$$\omega_r(t) = \frac{1}{\sqrt{2}}\Omega(t). \tag{2.318}$$

Since the phase volume of beam (2.315) is invariant, radius of matched beam is inversely proportional $\sqrt{\omega_r \gamma}$. Hence

$$R \doteqdot \frac{1}{\sqrt{\Omega(t)\gamma(t)}}$$

and with an increase in the energy of particles the size/dimension of beam will grow/rise. This increase/growth is very considerable.

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The frequency of small longitudinal vibrations is proportional $\gamma^{-1} p_s^{-1/2}$ (1.89); therefore

$$R \doteqdot p_s^{1/4},$$

or in nonrelativistic approximation/approach $R \div W_s^{1/8}$. For example, for the linear accelerator of protons I-100 [14] an increase in the radius of matched beam at the entire length of accelerator would comprise

$$\frac{\bar{R}_K}{\bar{R}_0} = \left(\frac{100}{0.7}\right)^{1/8} \approx 1.9.$$

Even in the ideal channel without the errors the amplitude of radial oscillations toward the end of the accelerator would grow almost two times.

2. Frequency of radial of synchronous particle is constant along axis of accelerator. Let us assume

$$\omega_L(0) = \Omega(0), \qquad (2.319)$$

then

$$\omega_r = \frac{\Omega(0)}{\sqrt{2}}. \qquad (2.320)$$

A radius of matched beam will remain in the nonrelativistic approximation/approach constant along the axis of accelerator, and the frequency of Larmor precession will relatively slowly descend

$$\omega_L(t) = \frac{\Omega(0)}{\sqrt{2}} \sqrt{1 + \left[\frac{\Omega(t)}{\Omega(0)}\right]^2}. \qquad (2.321)$$

The matched beam of relativistic particles retains a constant radius along the axis of accelerator when $\omega_r\gamma = \mathrm{const}$. If is satisfied condition (2.319), then

$$\omega_L(t) = \frac{\Omega(0)}{\sqrt{2}} \cdot \frac{\gamma(0)}{\gamma(t)} \sqrt{1 - \frac{p_s(0)}{p_s(t)}}. \qquad (2.322)$$

In relativistic region of energies the induction of focusing field with the retention/preservation/maintaining of the sizes/dimensions of beam descends considerably more rapid than in the nonrelativistic (2.321).

The three-dimensional/space period of radial oscillations for the particles with zero initial moment/torque L_r is determined by expression (2.302). Let L - period of the arrangement of solenoids. If L is commensurated with L_r or more than L_r then should be considered the periodic structure of focusing field, converting/transferring from the equations with the constant coefficients for eq tions with periodic coefficients [60, 72] and calculating the appropriate functions of Floquet or the matrix/die of period L. But if $L \ll L_r$ then nonuniformity in a number of ampere turns along the axis can lead only to a small "vibration" of trajectories by disregarding which let us reduce the problem with the aid of the smooth approximation/approach to focusing examined above in the stationary field. This case is feasible, in particular, in the system with drift tubes, if solenoids are placed into each tube. Actually/really, let L=βλ.

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Then

$$\frac{L}{L_r}=\frac{\beta\lambda}{2\pi\beta c/\omega_r}=\frac{\omega_r}{\omega}$$

or taking into account expressions (2.318), (2.320)

$$\frac{L}{L_r}=\frac{1}{\sqrt{2}}\cdot\frac{\Omega}{\omega}\ll 1.$$

The effective magnetic field on the axis of channel, which is determining the frequency of Larmor precession, is equal

$$B=\mu_0 nI,$$

where to $\mu_0=4\pi\cdot10^{-7}$ H/m; n - average number of turns, per unit of the length of channel. Let n_0 - number of turns per the unit of the length of solenoid. Let us assume that the solenoid completely fills the length of drift tube. Then

$$\frac{n_0}{n}=\frac{1}{1-\alpha},$$

where α - coefficient of clearance. The magnetic field on the axis of solenoid, which is determining ampere-turns n_0I, is equal

$$B_0=\frac{B}{1-\alpha}=\mu_0 n_0 I. \qquad (2.323)$$

In each solenoid is scattered the power, equal to

$$P\approx 2\cdot10^{12}\frac{\varrho(2a+b)}{bf}L_cB_0^2, \qquad (2.324)$$

where L_c - length of solenoid; a - radius of the internal cavity of solenoid; b - thickness of the lap; ρ - specific resistance of lead/duct, $\Omega\cdot$m; f - duty factor. As can be seen from expression (2.324), isolatable power with given B_0 linearly depends on a radius of aperture a in contrast to the dissipated power in the quadrupoles where the latter with the assigned field gradient is proportional to the fourth degree of aperture.

One of the deficiencies/lacks in the focusing in the longitudinal magnetic field is the appearance of large forces of interaction between the solenoids. Let us examine, for example, force between two single-layer solenoids of radius R with numbers of turns N_1, N_2, distant behind each other at a distance g. Let the lengths of solenoids L_1, L_2 substantially exceed their radius: $L_{1,2} \gg R$. The mutual inductance of such solenoids is determined by the expression

$$M = \frac{\pi\mu_0 N_1 N_2}{2L_1 L_2} R^2 \left(\sqrt{R^2 - g^2} - g\right). \qquad (2.325)$$

Interaction energy of the solenoids

$$W = M I_1 I_2. \qquad (2.326)$$

Substituting expressions (2.323), (2.325) into equality (2.326), we have

$$W = 1{,}25 \cdot 10^6 \left(\sqrt{R^2 - g^2} - g\right) R^2 B_0^2.$$

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The force of interaction between the solenoids exists

$$F = -\frac{dW}{dg} H.$$

Since 1 kg=9.8 N, the force of interaction in the kilograms is equal to

$$F = 1{,}28 \cdot 10^5 \left(1 - \frac{g}{\sqrt{R^2 - g^2}}\right) R^2 B_0^2. \qquad (2.327)$$

In contrast to the quadrupole lenses where the fields of adjacent lenses are virtually divided, in the solenoids the fields strongly engaged, that also leads to the large forces of electrodynamic interaction. These forces completely are applied to the end solenoids. For drift tubes, which have adjacent tubes from both sides, the forces to a considerable degree are balanced and there remains only difference force, connected with the dissimilar gap lengths between tubes. In the long resonator, loaded with drift tubes, adjacent clearances differ along the length to value

$$\Delta g = \alpha\lambda\Delta\beta.$$

Net force, equal to a difference in the forces, that act from the side of adjacent solenoids, there is

$$\Delta F = \frac{dF}{dg}\Delta g.$$

or

$$\Delta F = -1.28\cdot 10^5 \frac{R^4 B_0^2}{(R^2+g^2)^{3/2}}\Delta g \; \kappa z. \qquad (2.328)$$

For calculating the allowances for magnetic and structural/design errors in the focusing channel it is expedient to return to Cartesian coordinates, since in these coordinates equations of motion are linear. Let us first examine the character of trajectories in the Cartesian coordinates. Constants of motion (2.297) and (2.306) in the Cartesian coordinates take the form

$$M = x\frac{dy}{dt} - y\frac{dx}{dt} - \omega_L(x^2 - y^2) \qquad (2.329)$$

$$I = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + (\omega_L^2 - \omega_r^2)(x^2 + y^2) - 2\omega_L\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right). \qquad (2.330)$$

We will search for the solutions of equations of motion (2.295) in the form

$$x = A\sin(\omega_j t - \Theta);$$
$$y = A\cos(\omega_j t + \Theta).$$

Substituting these solutions into expression (2.295), we obtain the characteristic equation

$$\omega_j^2 - 2\omega_L\omega_j - \frac{1}{2}\Omega_\varphi^2 = 0.$$

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Frequency ω_i has two values

$$\omega_1 = \omega_L - \omega_r;$$
$$\omega_2 = \omega_L - \omega_r.$$

The general solution of equations of motion (2.295) takes the form

$$\begin{aligned} x &= A_1 \sin(\omega_1 t - \Theta_1) - A_2 \sin(\omega_2 t - \Theta_2); \\ y &= A_1 \cos(\omega_1 t - \Theta_1) - A_2 \cos(\omega_2 t - \Theta_2). \end{aligned} \qquad (2.331)$$

It is expressed the amplitude of oscillations A_1, A_2 through the appropriate values of constants of motion M, I, after substituting expressions (2.231) into integrals (2.329), (2.330):

$$\begin{aligned} A_1 &= \frac{1}{2\omega_r}\sqrt{1 - 2\omega_r M}; \\ A_2 &= \frac{1}{2\omega_r}\sqrt{1 - 2\omega_r M}. \end{aligned} \qquad (2.332)$$

They are of interest of the projection of trajectories on the transverse plane XOY. Let us examine two cases: accelerating field is absent $(\Omega_\varphi = 0)$ and particles are accelerated by high-frequency field $(\Omega_\varphi = \neq 0)$.

Case the first $\Omega_\varphi = 0$.

In this case $\omega_r = \omega_L$; $\omega_1 = 2\omega_L$; $\omega_2 = 0$ and solutions (2.331) take the form

$$\begin{aligned} x &= A_2 \sin\Theta_2 + A_1 \sin(2\omega_L t + \Theta_1); \\ y &= A_2 \cos\Theta_2 + A_1 \cos(2\omega_L t + \Theta_1). \end{aligned} \qquad (2.333)$$

Each particle on plane XOY rotates with an angular velocity of $2\omega_L$

around the point with coordinates $x_0=A_2\sin\Theta_2$, $y_0=A_2\cos\Theta_2$ (Fig. 2.27). The position of center and radius of gyration depend on initial conditions. The maximum distance of each particle from the axis is equal

$$r_{макс}=A_1+A_2.$$

According to expressions (2.329), (2.330),

$$I-2\omega_L M=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2, \qquad (2.334)$$

so that radius of gyration is equal to

$$A_1=\frac{1}{2\omega_L}\sqrt{\dot{x}^2+\dot{y}^2}. \qquad (2.335)$$

With an increase in the magnetic field the radius of gyration decreases inversely proportional to the value of field. For the particles with the zero initial moment of momentum $A_1=A_2$. The trajectories of these particles run through the origin of coordinates, in other words, through the axis relative to which M=0.

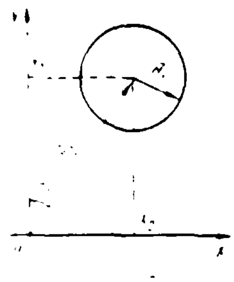

Fig. 2.27.

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Since any inscribed angle is two times lower than the central angle, which rests on the same arc, the angular rate of rotation relative to the origin of coordinates two times less than $2\omega_L$, i.e., is equal to the frequency of Larmor precession.

Second case $\Omega_\varphi \neq 0$.

In this case there is a physically chosen axis - axis of accelerating field. Each particle rotates with an angular velocity of $\omega_1 < 2\omega_L$ around certain axis which in turn, relatively slowly (with a speed of $\omega_2 = \omega_L - \omega_r$) rotates around the axis of accelerating field.

For the particles, which achieve maximum removal/distance from the axis, in both cases of M=0, $A_1 = A_2$ and, according to expressions

(2.307), (2.332).

$$A_1 = A_2 = \frac{R}{2}.$$

Thus, particle trajectories, which do not possess initial rotation and which achieve maximum removal/distance from the axis, take the form

$$\begin{aligned} x &= \frac{R}{2}\{\sin(\omega_1 t + \Theta_1) - \sin(\omega_2 t - \Theta_2)\}; \\ y &= \frac{R}{2}\{\cos(\omega_1 t - \Theta_1) - \cos(\omega_2 t - \Theta_2)\}. \end{aligned} \quad (2.336)$$

Let us establish/install now allowances on an error in the focusing field. The rotation of solenoids around the longitudinal axis in view of the axial symmetry of field does not cause the disturbances/perturbations of motion. Allowance for the parallel displacement of solenoids relative to longitudinal axis is caused only by the inclination/slope of field lines in the gaps/intervals between the solenoids and in the first approximation, it is possible not to consider it. Basic errors in the focusing field, which call the essential disturbance/perturbation of transverse vibrations, are connected with the inclinations/slopes of solenoids from the rating. For evaluating the allowances let us examine only those particles which achieve maximum removal/distance from the axis of channel. According to expressions (2.307, (2.330),

$$\omega_r^2 R^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 - \frac{1}{2}\Omega_\varphi^2 (x^2 + y^2). \quad (2.337)$$

Let us differentiate expression (2.337), let us square and it is averaged on all disturbances/perturbations and on all phases of transverse vibrations of particles. We will consider random disturbances in all solenoids equally probable and independent variables. As it is possible to show, utilizing expressions (2.336) and taking into account the equal probability of all values of the phases of transverse vibrations,

$$\langle x\rangle = \langle y\rangle = \frac{R}{2};$$

$$\left\langle \frac{dx}{dt}\right\rangle = \left\langle \frac{dy}{dt}\right\rangle = \frac{R}{2}\sqrt{\omega_L^2-\omega^2}.$$

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For the root-mean-square disturbance/perturbation of amplitude we obtain the expression

$$\langle \delta R\rangle^2 = \frac{1}{\omega_r^2}\left[1+\left(\frac{\Omega_\varphi}{2\omega_r}\right)^2\right]\langle \delta \dot{x}\rangle^2 + 2\left(\frac{\Omega_\varphi}{2\omega_r}\right)^4\langle \delta x\rangle^2.$$

Disturbances/perturbations of coordinate give the contribution to an increase in the amplitude only in the presence of accelerating field. Subsequently let us examine only channel with the constant frequency of radial (2.320). In the larger part of this channel

$$\left(\frac{\Omega}{2\omega_r}\right)^2 = \frac{1}{2}\left(\frac{\Omega}{\Omega_0}\right)^2 \ll 1,$$

therefore

$$\langle \delta R\rangle \approx \frac{1}{\omega_r}\langle \delta \dot{x}\rangle.$$

Let there be in the channel N_c solenoids. Then toward the end of the channel the root-mean-square increase of amplitude is equal

$$\langle \Delta R\rangle = \frac{\sqrt{N_c}}{\omega_r}\langle \delta \dot{x}\rangle. \qquad (2.338)$$

Let us connect $\delta\dot{x}$ in one solenoid with the slant of solenoid to values Δl in each of the coordinate planes and with the instability of field ΔB/B. Taking into account that in the larger part of channel

$\left(\frac{\Omega}{\Omega_n}\right)^2 \ll 1$, let us disregard/neglect the difference between values ω_r and ω_L, which considerably simplifies estimation. Then $\omega_1 \approx 2\omega_L$, $\omega_2 \approx 0$ and from expression (2.336) we have

$$\frac{dx}{dt} = \omega_L R \cos(2\omega_L t - \Theta_1);$$
$$\frac{dy}{dt} = -\omega_L R \sin(2\omega_L t - \Theta_1).$$

Transverse outlet velocity from the solenoid is connected with the inlet velocity with the relationship/ratio

$$\left(\frac{dx}{dt}\right)_{L_c} = \left(\frac{dx}{dt}\right)_0 \cos K + \left(\frac{dy}{dt}\right)_0 \sin K, \qquad (2.339)$$

where L_c - length of solenoid, and

$$K = L_c \frac{eB}{p_s} \qquad (2.340)$$

characterizes the "hardness" of solenoid. Let α_x, α_y angles of the slope of the axis of solenoid to the axis of channel in each coordinate plane. In the presence of errors α_x, α_y, ΔK, we have

$$\dot{x}_{L_c} = v\alpha_x + (\dot{x}_0 - v\alpha_x)\cos(K + \Delta K) + (\dot{y}_0 - v\alpha_y)\sin(K + \Delta K). \quad (2.341)$$

Deducting expression (2.339) from equality (2.341), in the first approximation, we obtain

$$\delta\dot{x} = (1 - \cos K)\, v\alpha_x - v\alpha_y \sin K - (\dot{x}_0 \sin K - \dot{y}_0 \cos K)\,\Delta K.$$

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Is averaged the square of the disturbance/perturbation of transversing speed. Taking into account that

$$\langle\alpha_x\rangle = \langle\alpha_y\rangle = \frac{1}{L_c}\langle\Delta l\rangle;$$
$$\overline{\dot{x}_0 \dot{y}_0} = 0; \qquad \frac{\Delta K}{K} = \frac{\Delta B}{B}.$$

we obtain the equality

$$\langle \dot{\delta x}^2 \rangle = \frac{c^2 K^2}{4L_c^2}\left[4\left(\frac{\sin\frac{K}{2}}{\frac{K}{2}}\right)^2 \langle \Delta l^2 \rangle - \frac{1}{2}K^2R^2 \left\langle \left(\frac{\Delta B}{B}\right)^2 \right\rangle\right].$$

Hence

$$\langle \Delta R \rangle = \sqrt{N}\left[4\left(\frac{\sin\frac{K}{2}}{\frac{K}{2}}\right)^2 \langle \Delta l^2 \rangle \frac{1}{2}K^2R^2 \left\langle \left(\frac{\Delta B}{B}\right)^2 \right\rangle\right]^{1/2}. \quad (2.342)$$

First term in brackets (2.342) characterizes the external disturbance/perturbation, which leads to the beam displacement as a whole, and the second connected with the parametric disturbance/perturbation, depending on the amplitude of oscillations.

Let us examine the possible parameters of the focusing channel with the longitudinal magnetic field for the linear accelerator of protons. As the initial data let us accept the parameters of the proton accelerator I-100 [14]: λ=2 m; $W_\lambda = 2.7\cdot 10^{-3}$; $W_0 = 700$ keV; $W_к = 100$ MeV. At the entrance $\Omega(0)/\omega=9.35\cdot10^{-2}$. Let us assume in accordance with expression (2.319) $\omega_L(0)=\Omega(0)$ and let us accept $\omega_r = \text{const}$ at the length of accelerator. According to formula (2.320), $\omega_r = 6.60\cdot10^{-2}\cdot\omega$. At the output of the accelerator [see expression (2.321)] we have $\omega_L = 0.8\Omega(0)$. From formula (2.294) it follows: $B_{нач} = 18\,400$ G; $B_{кон} = 14\,700$ G. With the coefficient of clearance α=0.25 the fields, created by solenoids in drift tubes, must be equal to: $B_{0\,нач} = 24\,500$ G; $B_{0\,кон} = 19\,600$ G.

Let us accept the phase volume of beam equal to $V_n = 0.1$ cm·mrad. With this phase volume a radius of matched beam in accordance with formula (2.315) will be R=0.22 cm. As it will be shown in the following chapter, during the numerical estimation of the matched initial conditions for the strong-focusing channel of the same accelerator, in the system of quadrupole lenses $R_\Phi = 0.35$ cm; $R_\lambda = 0.15$ cm. Hence it is apparent that during the optimum identification of parameters of both focusing channels the size/dimension of beam in the longitudinal magnetic field proves to be smaller than the maximum size of beam in the strong-focusing channel.

The hardness of solenoids (2.340) in the beginning of accelerator is equal to K=1.15. We have $\frac{\sin K/2}{K/2} = 0.95 \approx 1$. If we assign $\frac{\Delta B}{B} = 1\%$, then the divergences of field from the rating in effect will not agitate trajectories. The basic perturbing factor is connected with the inclination/slope of the axes of solenoids. From formula (2.342) we will obtain

$$\langle \Delta R \rangle \approx 2 \sqrt{N_c} \langle \Delta l \rangle. \qquad (2.343)$$

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Analogous relationship/ratio for the strong-focusing channel took the form (see S 2.8)

$$\langle \Delta A \rangle \approx 0{,}83 \sqrt{N_\Phi}.$$

A number of solenoids is equal to the doubled number of periods of focusing field. An identical increment in the amplitude in both systems of focusing we will obtain with $\Delta l = 0.3$ mm. This allowance completely can be returned to the mechanical adjustment of the axes of drift tubes, so that it proves to be six times of wider than the appropriate allowance with the strong focusing. Thus, the adjustment of drift tubes with the solenoids considerably is facilitated in comparison with the adjustment of the tubes, which contain quadrupole lenses.

For evaluating the dynamic forces, experienced/tested by drift tubes, and dissipated power let us assign the following sizes/dimensions, entering formulas (2.324), (2.327), (2.328): $a = 1$ см; $b = 5$ см; $g = 2$ см; $R = 3.5$ см. Then the force, which acts on the end solenoid, is F=470 kg, and net force, which acts on internal solenoids, ΔF~~24 kg. Let us accept the duty factor of window with copper equal to f=0.2. In this case the first solenoid of accelerator isolates power $P_{нач} = 88$ kW, and the latter - $P_{кон} = 600$ kW. The supply of solenoids by direct current without the use/application of a deep cooling in this case virtually is eliminated, hook as it is impossible to lead from drift tubes the power indicated.

Accelerator I-100 works by narrow pulses, which makes it possible to utilize a pulse supply of magnetic lenses. With the pulse supply the dissipated power can be lowered more than 100 times.

Therefore let us examine the possibility of the supply of solenoids by pulse current. This analysis is interesting and because the general/common/total relationships/ratios remain valid also with the pulse supply of quadrupole lenses. Let the pulse current, which feeds lens, be obtained by resonance capacitor discharge through the valve/gate to the winding of lens. Let us assign two initial values: with the duration of the half-wave of current T and with maximum voltage across capacitor V. The inductance of winding is equal to

$$L = kN^2, \qquad (2.344)$$

where N - number of turns in the winding, and coefficient k depends on the geometry of winding. In particular, for the solenoid we have

$$k = \frac{A}{D + L_c}$$

(L_c - the length of solenoid). Values A, D depend on a radius of the internal cavity a and the thickness of the lap b;

$$A \approx \frac{9}{4\pi}\mu_0\left(a - \frac{b}{2}\right)^2; \quad D \approx \frac{1}{3}(a + 4b).$$

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During the capacitor discharge C to the inductance of winding L is fulfilled relationship/ratio $CV^2=LI^2$, moreover $T = \pi\sqrt{LC}$. Hence taking into account expression (2.344) we obtain

$$T = \frac{\pi k N}{V} NI.$$

According to expression (2.323),

$$NI = \frac{1}{\mu_0} B_0 L_c.$$

Thus,

$$N = \mu_0 \frac{VT}{\pi B_0} \cdot \frac{1}{kL_c} . \qquad (2.345)$$

The amplitude of coil current is equal to

$$I = \frac{\pi B_0^2}{\mu_0^2 V T} kL_c^2. \qquad (2.346)$$

Value of the discharge capacity

$$C = \frac{\pi B_0^2}{\mu_0^2 V^2} kL_c^2. \qquad (2.347)$$

Hence it is apparent that at the given values of T, V, B_0 and the geometry of windings a number of turns, the amplitude of current and the value of discharge capacity for each lens are determined unambiguously. For the majority of drift tubes D L_c; then $kL_c \approx A = \mathrm{const.}$ so that a number of turns in different solenoids does not depend on their length. Amplitude of current and capacitance value directly proportional to the length of each solenoid. An increase in the magnetic field decreases a number of turns and respectively increases current and discharge capacity.

The permissible working stress/voltage for the solenoids, placed within drift tubes, can be accepted equal to V=1.5 kV. Let the half-period of discharge be T=600 μs; this provides the necessary flat/plane part of the current pulse for the proton beams with a duration of 10-40 μs. In the parameters indicated the amplitude of current will range from 0.9 in the beginning of accelerator to 12.2

kava end/lead. Necessary capacitance value is changed from 113 in the beginning to 1560 μF at the end. The available capacity of capacitors/condensers will be 134000 μF. It is obvious that the use of a similar power-supply system for the proton linear accelerator to the energy 100 MeV is inexpedient.

Thus, in spite of the explicit advantages of the system of focusing by longitudinal magnetic field, the large dynamic forces, which appear between drift tubes, and the difficulties of the realization of heat withdrawal or creation of pulse supply force at present to turn from the use of this system to proton linear accelerators on the middle of energy at the wavelength of accelerating field 1.5-2 m.

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An increase in the wavelength of accelerating field with the fixed/recorded value of specific acceleration proportionally decreases, according to expression (2.304), the minimally allowed value of Larmor frequency and respectively the induction of focusing field. In this case power consumption per the unit of the length of accelerator and the dynamic forces between the solenoids descend inversely proportional to square the wavelengths.

§2.10. High-frequency quadrupoles.

Longitudinal and lateral stabilities of particles in the linear accelerator without the external focusing fields are incompatible only when accelerating field is axial-symmetrical traveling wave. In §2.1 it is explained that at the sufficiently low absolute values of synchronous phase the accelerating clearances between drift tubes focus particles. As long ago as 1953 Ya. B. Faynberg [73, 74] showed that reaching/achievement of the simultaneous stability of longitudinal and transverse vibrations possibly also in the case when synchronous phase periodically reverses the sign along the axis of accelerator.

In 1956 V. V. Vladimirskiy [34] proposed the method of guaranteeing longitudinal and lateral stability due to the failure of the axial symmetry of high-frequency field in the accelerating clearances. In the form of an example work [34] examines the introduction of "horns" to the accelerating clearance (Fig. 2.28a). Horns create the electric field whose transverse components are analogous to the field of electrostatic quadrupole lens. The longitudinal component of accelerating field is utilized for the particle acceleration, and transverse components - for the focusing. A deficiency/lack in a similar construction/design is the decrease in dielectric strength, caused by the introduction of horns to the accelerating clearance.

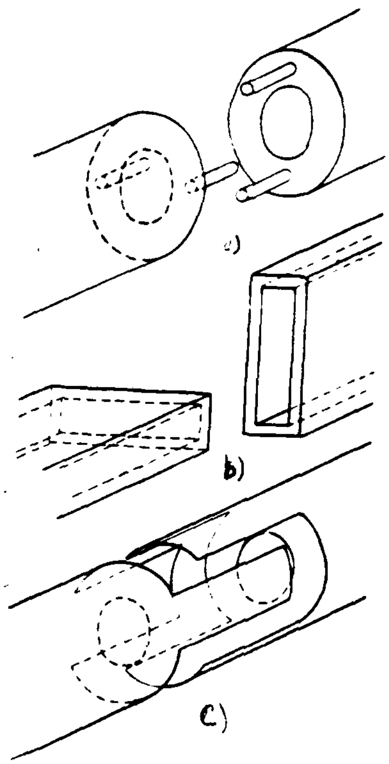

Fig. 2.28.

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V. A. Teplyakov [35, 36] investigated the system, deprived of this deficiency/lack. He proposed to make aperture opening/aperture in drift tube not circular, but rectangular, moreover each subsequent tube to turn on 90° by relatively by preceding/previous (see Fig. 2.28b). However, such a idea was expressed later by Fer and Lapostoll

[37]. One additional version, proposed by V. A. Teplyakov, it is shown in Fig. 2.28c; the form of horns in Fig. 2.28c it is changed in comparison with the form in Fig. 2.28a how is reached the best configuration of the focusing components of field [75].

Let us examine field in the clearance with the more general case of symmetry, than axial. Let the potential of field have two planes of symmetry XOZ and YOZ:

$$U(x, y, z) = U(-x, y, z) = U(x, -y, z). \qquad (2.348)$$

the transition from the point with coordinate z to the point with the coordinate - z with the simultaneous rotation on 90° in the transverse plane reversing the sign of potential, without changing its value

$$U(x, y, z) = -U(-y, x, -z). \qquad (2.349)$$

Point z=0 - the geometric center of clearance. The conditions of symmetry (2.348), 2.349) satisfy fields in all clearances, shown in Fig. 2.28.

We will be restricted to the nonrelativistic region of energies. Then the equations of motion of particle in the accelerating gap take the form

$$\begin{aligned} \frac{d^2x}{dt^2} &= \frac{e}{m_0} E_x \cos\omega t; \\ \frac{d^2y}{dt^2} &= \frac{e}{m_0} E_y \cos\omega t; \\ \frac{d^2z}{dt^2} &= \frac{e}{m_0} E_z \cos\omega t. \end{aligned} \qquad (2.350)$$

Subsequently let us examine the motion of paraxial particles. Let us expand the transverse components of field in the power series

$$E_{x,y}(x, y, z) = E_{x,y}(0, 0, z) + \frac{\partial E_{x,y}}{\partial x}(0, 0, z)\,x + \frac{\partial E_{x,y}}{\partial y}(0, 0, z)\,y + \ldots$$

and let us hold down/retain the linear terms of resolution. Then equations (2.350) for the transverse coordinates prove to be linear. From the conditions of symmetry (2.348) we have

$$E_x(0, 0, z) = E_y(0, 0, z) = 0;$$
$$\frac{\partial E_x}{\partial y}(0, 0, z) = \frac{\partial E_y}{\partial x}(0, 0, z) = 0. \qquad (2.351)$$

Equalities (2.351) follow directly from those considerations, what derivative of even function - function is odd.

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From the condition of symmetry (2.349) we have

$$\frac{\partial E_x}{\partial x}(0, 0, z) = -\frac{\partial E_y}{\partial y}(0, 0, -z); \qquad (2.352)$$
$$\frac{\partial E_z}{\partial z}(0, 0, z) = -\frac{\partial E_z}{\partial z}(0, 0, -z). \qquad (2.353)$$

For simplification in further recordings all functions, undertaken on the axis, f(0, 0, z) let us designate f(z). The equations of transverse motion (2.350) can be now represented in the form

$$\frac{d^2x}{dt^2} = \frac{e}{m_0}\cdot\frac{\partial E_x}{\partial x}(z)\cos\omega t\cdot x;$$
$$\frac{d^2y}{dt^2} = \frac{e}{m_0}\cdot\frac{\partial E_y}{\partial y}(z)\cos\omega t\cdot y. \qquad (2.354)$$

Let us introduce the function

$$G(x,\ y,\ z)=\frac{1}{2}\left[\frac{\partial E_x}{\partial x}(x,\ y,\ z)-\frac{\partial E_y}{\partial y}(x,\ y,\ z)\right].\qquad(2.355)$$

Since the derivatives of the components of field are connected with the general/common/total relationship/ratio

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}=-\frac{\partial E_z}{\partial z},$$

then

$$\begin{aligned}\frac{\partial E_x}{\partial x}&=-\frac{1}{2}\cdot\frac{\partial E_z}{\partial z}+G;\\ \frac{\partial E_y}{\partial y}&=-\frac{1}{2}\cdot\frac{\partial E_z}{\partial z}-G.\end{aligned}\qquad(2.356)$$

In the axisymmetric field G(x, y, z)≡0. Equalities (2.356) generalize expressions for forces (2.40) for a broader class of symmetries (2.348), (2.349). First terms in the right sides of equalities (2.356) cause the defocusing of particles in the accelerating clearances. Second terms, which are absent with the axial symmetry, can the principle provide strong focusing. We will consider the stability conditions of longitudinal vibrations as those carried out and let us find the requirements, presented to function G(x, y, z), which ensure the stability of transverse vibrations.

Let us expand functions $\frac{\partial E_z}{\partial z}(z)$ and G(z) in Fourier series in the period of the accelerating structure L. Function $\frac{\partial E_z}{\partial z}(z)$, according to expression (2.353), odd.

$$\frac{\partial E_z}{\partial z}(z)=\sum_{n=1}^{\infty}A_n\sin\frac{2\pi n}{L}z.\qquad(2.357)$$

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Function G(z), as this follows from equalities (2.352), (2.355), even

$$G(z)=\sum_{n=0}^{\infty} B_n \cos \frac{2\pi n}{L} z. \qquad (2.358)$$

Let us assume that the accelerating clearance on its action is equivalent to thin lens. Then particle displacements at the gap length are not changed, but transversing speeds obtain the increases

$$\begin{aligned} \Delta \frac{dx}{dt} &= \frac{e}{m_0} x \int_L \frac{\partial E_x}{\partial x}(z) \cos \omega t(z) \frac{dz}{v}; \\ \Delta \frac{dy}{dt} &= \frac{e}{m_0} y \int_L \frac{\partial E_y}{\partial y}(z) \cos \omega t(z) \frac{dz}{v}. \end{aligned} \qquad (2.359)$$

Let us disregard/neglect the effect of a change in the particle speed at the gap length, after assuming

$$t(z)=\frac{z}{v}+\frac{1}{\omega}\varphi. \qquad (2.360)$$

Let

$$b_1=-\frac{e}{2m_0 v}\int_L \frac{\partial E_z}{\partial z}(z) \cos \omega t(z)\, dz;$$

$$b_2=\frac{e}{m_0 v}\int_L G(z) \cos \omega t(z)\, dz.$$

Then, substituting expressions (2.356) in integrals (2.359), we obtain

$$\begin{aligned} \Delta \frac{dx}{dt} &= b_x x; \\ \Delta \frac{dy}{dt} &= b_y y. \end{aligned} \qquad (2.361)$$

where

$$b_x=b_1+b_2; \quad b_y=b_1-b_2. \qquad (2.362)$$

Let us introduce series/rows (2.357), (2.358) into the expressions for b_1, b_2. Taking into account formula (2.360)

$$b_1 = \frac{e\lambda k}{4m_0c} A_k \sin\varphi;$$

$$b_2 = \frac{e\lambda k}{2m_0c} B_k \cos\varphi.$$

where k - multiplicity of the period of the accelerating structure; $L=k\beta\lambda$. As a result for the refractive indices we obtain the following expressions

$$\begin{aligned} b_x &= \frac{e\lambda k}{2m_0c}\left(\frac{A_k}{2}\sin\varphi - B_k\cos\varphi\right); \\ b_y &= \frac{e\lambda k}{2m_0c}\left(\frac{A_k}{2}\sin\varphi - B_k\cos\varphi\right). \end{aligned} \qquad (2.363)$$

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Let the period of focusing field consist of two periods of accelerating structure (S=2L) and contains two accelerating clearances. In each following clearance the electrodes are turned on 90° relative to preceding/previous, so that planes XOZ and YOZ transpose. The matrix/die of the period of focusing field can be represented by the product

$$T=H_{1/2}\bar{\Gamma}H\Gamma H_{1/2}.$$

Matrix/die T corresponds to the period which begins and is terminated in the middle of the idle gaps/intervals between the clearances; H - matrix/die of the idle gap/interval

$$H=\begin{pmatrix}1 & \frac{k\lambda}{c}\\ 0 & 1\end{pmatrix}.$$

In this case it is assumed that the high-frequency field is concentrated in the thin clearance. Γ and $\bar{\Gamma}$ — matrix/die of the adjacent clearances

$$\Gamma=\begin{pmatrix}1 & 0\\ b_x & 1\end{pmatrix};\quad \bar{\Gamma}=\begin{pmatrix}1 & 0\\ b_y & 1\end{pmatrix}.$$

The multiplication of matrices/dies gives

$$\cos\mu=1+\frac{k\lambda}{c}(b_x+b_y)+\frac{1}{2}\left(\frac{k\lambda}{c}\right)^2 b_x b_y.$$

Let us replace refractive indices with their expressions (2.363):

$$\cos\mu=1+\frac{ek^2\lambda^2}{2\mathscr{E}_0}A_k\sin\varphi\left(1+\frac{ek^2\lambda^2}{16\mathscr{E}_0}A_k\sin\varphi\right)-$$
$$-\frac{1}{2}\left(\frac{ek^2\lambda^2}{2\mathscr{E}_0}A_k\right)^2\left(\frac{B_k}{A_k}\right)^2\cos^2\varphi. \qquad (2.364)$$

Let us examine the components/terms/addends, entering the right side of equality (2.364). Regarding

$$A_k = \frac{2}{L}\int_L \frac{\partial E_z}{\partial z}(z)\sin\frac{2\pi kz}{L}\,dz.$$

Let us take this integral in parts:

$$A_k = \frac{2}{L}\,E_z(z)\sin\frac{2\pi kz}{L}\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}} - \frac{4\pi k}{L^2}\int_L E_z(z)\cos\frac{2\pi kz}{L}\,dz.$$

The first term at both ends/leads of the interval of integration becomes zero. The integral, entering the second term, according to expression (1.11), takes the form

$$\int_L E_z(z)\cos\frac{2\pi z}{\beta\lambda}\,dz = E_0 TL,$$

where E_0 - middle field; T - factor of transit time.

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Hence

$$A_k = -4\pi\frac{E_0 T}{\beta\lambda}.$$

Since $A_k<0$ and, therefore, for the particles, close to the synchronous, $A_k\sin\varphi>0$. second term in the right side of equation (2.364) determines the defocusing action of high-frequency field. Third component/term/addend decreases $\cos\mu$ and, therefore, is described the focusing action of field in the clearance. Third component/term/addend depends on the phase of particle and middle field, which in principle differs the focusing in question by external quadrupole fields. Into all members of expression (2.364) enters the combination of the parameters

$$\eta = -\frac{ek^2\lambda^2}{2\mathcal{E}_0}A_k.$$

Replacing A_k, we obtain

$$\eta = \pi k \frac{\Delta W_{макс}}{W_s}.$$

where W_s — current energy of synchronous particle; $\Delta W_{макс}$ — maximally possible energy gain of particle in the period of the accelerating structure

$$\Delta W_{макс} = eE_0LT.$$

Coefficient η can be also determined by the equality

$$\gamma = -\eta \sin\varphi.$$

where γ - factor of defocusing (2.170). In further more conveniently to operate with coefficient η, since it does not depend on the phase of particle, which simplifies the comparison of the components/terms/addends in the right side of expression (2.364).

Regarding coefficient B_k it is equal to

$$B_k = \frac{2}{L}\int\limits_L G(z)\cos\frac{2\pi kz}{L}dz. \qquad (2.365)$$

Let E_g — average/mean amplitude value of the longitudinal component of high-frequency field at the length of the accelerating clearance g:

$$E_g = \frac{1}{g}\int\limits_L E_z(z)\,dz.$$

Then $gE_g = LE_0$. Expression (2.365) can be represented in the form

$$B_k = 2\frac{\int\limits_L G\cos\frac{2\pi z}{\beta\lambda}dz}{\int\limits_L G\,dz}\cdot\frac{1}{E_g}\int\limits_L G(z)\,dz\frac{E_0}{g}.$$

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Value

$$T_g = \frac{\int\limits_L G \cos \frac{2\pi z}{\beta\lambda}\, dz}{\int\limits_L G\, dz} \qquad (2.366)$$

is analogous to the factor of transit time and in the narrow accelerating gaps is close to unity just as T. For the tubes with the rectangular apertures in the first approximation,

$$\frac{T_g}{T} = \frac{\pi\alpha}{\mathrm{tg}\,\pi\alpha},$$

where α - coefficient of clearance (1.23). For horned tubes $T_g \approx T$, since function G(z), as function $E_z(z)$, is approximately constant at the gap length. The dimensionless quantity

$$K_g = \frac{1}{E_g} \int\limits_L G(z)\, dz \qquad (2.367)$$

characterizes the geometry of high-frequency field in the clearance. In the axisymmetric field $K_g = 0$. Parameter K_g is uniquely determined by the configuration of field in the clearance and does not depend on the length of the period of the accelerating structure, since integral (2.367) actually is taken on the section, occupied with field. During the calculation or in measurement K_g it is convenient to use the relationships/ratios

$$K_g = \frac{1}{E_g}\int_L \frac{\partial E_x}{\partial x}\,dz = -\frac{1}{E_g}\int_L \frac{\partial E_y}{\partial y}\,dz.$$

Thus,

$$B_k = \frac{2E_0}{g} K_g T_g.$$

Hence we obtain the ratio of coefficients, entering expression (2.364):

$$\frac{B_k}{A_k} = -\frac{K_g}{2\pi a}\cdot\frac{T_g}{T}.$$

Equality (2.364) can be written in the form

$$\cos\mu = 1 - \eta\sin\varphi\left(1 - \frac{1}{8}\eta\sin\varphi\right) - \frac{1}{2}\eta^2\left(\frac{K_g T_g}{2\pi a T}\right)^2\cos^2\varphi.$$

Parameter η is low in comparison with unity. As are shown detailed calculations, if are satisfied the conditions

$$\frac{1}{8}\eta^2\left(\frac{K_g T_g}{2\pi a T}\right)^2 \ll 1;\quad \frac{1}{64}\eta^2 \ll 1,$$

then in the term, determining defocusing, it is possible to disregard the component/term/addend, proportional η^2. Then

$$\cos\mu = 1 - \eta\sin\varphi - \frac{1}{2}\eta^2\left(\frac{K_g T_g}{2\pi a T}\right)^2\cos^2\varphi. \qquad (2.368)$$

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Let us require the retention/preservations/maintaining the stability of all trajectories in any phases of particles in the limits of separatrix. For this must be fulfilled the inequalities

$$1 - \eta\sin 2\varphi_s - \frac{1}{2}\eta^2\left(\frac{K_g T_g}{2\pi a T}\right)^2\cos^2 2\varphi_s < 1;$$

$$1 + \eta\sin\varphi_s - \frac{1}{2}\eta^2\left(\frac{K_g T_g}{2\pi a T}\right)^2\cos^2\varphi_s > -1.$$

From the first inequality follows

$$\frac{\operatorname{tg} 2\varphi_s}{\cos 2\varphi_s} < \frac{1}{2}\eta\left(\frac{K_g T_g}{2\pi a T}\right)^2. \qquad (2.369)$$

From the second inequality we have

$$\left(\frac{K_g T_g}{2\pi\alpha T}\right)^2 < 2\,\frac{2-\eta\,\sin\varphi_s}{\eta^2}. \quad (2.370)$$

Condition (2.369) determines the maximum permissible value/significance of synchronous phase φ_s with given ones η, α, K_g or the minimally allowed value of coefficient of asymmetry of field K_g with given ones φ_s, η, α. Condition (2.370) is given upper boundary for the coefficient of asymmetry K_g. Basic practical value/significance has condition (2.369), since to obtain high values K_g is difficult, and to have the low absolute values of synchronous phase is disadvantageous due to the decrease of capture region.

Work [37] gives the results of modeling of clearance with the rectangular aperture openings/apertures on the electrolytic bath in the approximation/approach of two-dimensional problem. According to the measurements of field distribution were calculated the coefficients of asymmetry K_g. Some results of measurements are given in Fig. 2.29. Here a - narrow side of rectangular aperture; b - wide side. As can be seen from graphs/curves, $K_g \leqslant 1$.

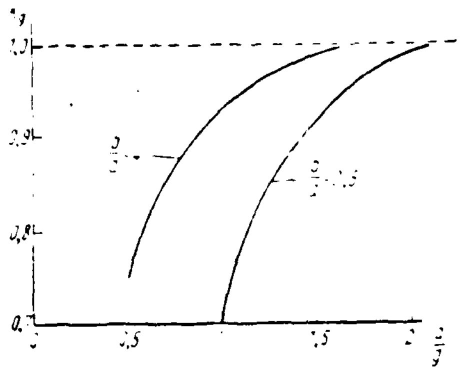

Fig. 2.29.

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Particle focusing by the high-frequency quadrupoles possesses advantages, which consist in large structural/design simplicity of drift tubes, possibility of decreasing the diameters of tubes (since within the tubes it is not necessary to place magnetic lenses), in the absence of the complicated and bulky equipment of the supply of lenses. However, in the short-wave proton accelerators (λ=1.5-2 m) focusing with the aid of the rectangular aperture openings/apertures leads to the inadmissible decrease of capture region. Thus, let us assume, $K_g = 1$; $\frac{T_g}{T} = 1$. Then with λ=2 m; α=0.25; k=1; β=0.04 and E_0=16 kV/cm we obtain $\eta = 0.53$ and condition (2.369) will lead to the inequality

$$\frac{|\mathrm{tg}\, 2\varphi_s|}{\cos 2\varphi_s} < 0{,}11,$$

or $|\varphi_s| < 3°$. This value is so low which is necessary to consider the effect of a change in gap velocity. Taking into account the effect of an increase in velocity φ_s it will increase approximately/exemplarily doubly. From an increase in the energy of particles parameter η falls, which decreases upper threshold φ_s. Hence it is apparent that in short-wave proton accelerator the effect of high-frequency focusing virtually is absent. Expansion of capture region with the particle focusing with the aid of the alternately oriented rectangular apertures can be achieved/reached by an increase in parameter η or decrease of the coefficient of clearance α. Since

$$\eta = \frac{2\pi k^2 T}{c^2} \cdot \frac{eE_0\lambda}{m_0\beta},$$

then increase of η is possible with: a reduction in the exit energy of particles; an increase in the wavelength of accelerating field; an increase in the energy gain per unit of the length of accelerator; the acceleration of polyvalent ions with the increased ratio e/m_0. Let us note that with the very low energies of particles and upon the high specific acceleration exactly becomes difficult the focusing by magnetic quadrupoles. If middle field and gap lengths are fixed/recorded, then $E_g \doteqdot \lambda$; $\frac{\eta}{\alpha^2} \doteqdot \lambda^3$. Therefore an increase in the wavelength makes it possible to substantially expand capture region with an insignificant increase in the field in the clearances. Focusing with the aid of the rectangular apertures proves to be most effective in the long-wave accelerator of polyvalent ions.

The coefficient of asymmetry K_g can be more than unity in the accelerating gaps with advanced electrodes (see Fig. 2.28c). V. A. Teplyakov obtained disregarding small clearances between electrodes advanced into the accelerating gap the following formula for the coefficient of the asymmetry

$$K_g = \frac{3\sqrt{3}}{\pi}\,\alpha\,\mathrm{tg}\,\pi\alpha\,\frac{I_0\left(2\pi\frac{r_0}{\beta\lambda}\right)}{I_2\left(2\pi\frac{r_0}{\beta\lambda}\right)}.$$

where r_0 - radius of drift tube (distance from the axis to the electrode).

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At the low values of the argument of the modified Bessel functions

$$I_0 \approx 1;\quad I_2 \approx \frac{\pi^2}{2}\left(\frac{r_0}{\beta\lambda}\right)^2$$

Hence

$$K_g = \frac{6\sqrt{3}}{\pi^3}\,\alpha\,\mathrm{tg}\,\pi\alpha\left(\frac{\beta\lambda}{r_0}\right)^2.$$

The focusing action of such clearances with α=const) and with the decrease of the diameter of drift tubes (which in turn, is connected with an increase in the gradient of the focusing components of field). Clearances with the advanced electrodes can solve the problem of particle focusing of high-energy during the guarantee of necessary dielectric strength.

From that presented it is evident that the beam focusing in the linear accelerators with the aid of the high-frequency quadrupoles in a number of cases is promising. This method of focusing requires further investigations and searches for the most optimum configurations of the accelerating structure.

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Chapter 3.

Transverse vibrations of particles in the beams with the high density of space charge.

§ 3.1. Formulation of the problem. Separation of variables in the equations of motion.

In the first chapters was examined the particle motion in the assigned applied fields (accelerating and focusing) without taking into account electrical interaction of particles. It was assumed that each particle moves in the manner that as if other charged/loaded particles in the channel there does not exist. This assumption is admissible, if the density of space charge in any beam section is sufficiently small. However, the estimation of the possibility to disregard/neglect the effect of Coulomb interaction of particles can be given only within the framework of the more general theory, in which this effect is considered.

"Single-particle" theory made it possible to determine the parameters of accelerator, in which is provided necessary stability of motion of particles. But even when current density in the accelerated beam is negligible, single-particle theory cannot answer many questions, which appear during the design of accelerators. Taking into account the phase volume of beam as the measure of the scatter of particles for position and velocities, we is exceeded the limits of single-particle theory. Introduction to the calculations of the value of the phase volume of beam made it possible to calculate the envelope of the trajectories of the collective of particles, i.e., to determine the sizes/dimensions of beam in the focusing channel and to answer a question about that, will pass this beam through the focusing channel without the essential loss of intensity. Was determined the optimum configuration of beam at the entrance of channel, i.e., were found the matching conditions of beam with the channel. It turned out that one of the most important characteristics of the focusing channel is its capacity.

Were explained the supplementary requirements for the sources of particles, connected with the fact that the phase volume of beam must not exceed the capacity of the focusing channel; an increase in the intensity of beam must be accompanied by an increase in the phase particle density.

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However, for solving the questions about the limiting current of the beam of the accelerated particles, about the identification of the parameters of the accelerator, designed for the acceleration of beam with the assigned intensity, about the agreement of intense beam with the channel and others of the theory presented it is insufficient. Are necessary the relationships/ratios, which consider interaction of the charged/loaded particles in the beam. Thus, further fundamental withdrawal/departure from the single-particle theory is connected taking into account the forces of interaction between the particles of beam, i.e., taking into account the electric fields of all charged/loaded particles.

The direct method of calculation of particle motion taking into account their electrical interaction consists in comprising as many equations, as particles in the beam, after introducing into the examination all two-body forces, and to solve equations together, after assigning the totality of initial conditions for the particles of the beam. It is obvious that the solution of this direct problem is virtually unrealizable, since a number of particles in the beam is too great. Therefore it is necessary to deal concerning the model of bundle, limiting a number of particles with the reasonable value in accordance with the storage capacity of electronic computer. This posing of the question proves to be already approximate. But also in this case the substantiation of sufficiently general/common/total conclusions/outputs requires obtainings of many particular solutions that it entails the expenditure of too long machine a time. Hence it is apparent that the possibilities of direct method are very limited.

Another method lies in the fact that the superposition of the fields of a large number of discrete/digital particles to replace with the field of the continuous space charge. Then is examined the motion of one particle in the assigned applied fields and in the field of the space charge of beam. After considering the distribution of initial conditions, i.e., the phase volume of beam, it is possible to find envelope of particles and to answer other interesting us questions by the methods, analogous to those described. However, this method requires correct approach. It is not possible to compose equation of motion, on the basis of which predetermined density distribution of charge, since the obtained solutions can prove to be by such, with which the assigned charge distribution (taking into account which it was determined solution) is not retained. Such solutions are internally contradictory and in the general case can lead to the quantitative and qualitative errors. Therefore charge distribution must be defined from the simultaneous equations of motion and field just as particle trajectory. Obtained from the totality of the equations of mechanics and electrodynamics the proper field of beam is called self-consistent. Self-congruent field causes this particle motion, what determines precisely this field distribution.

Are possible two formulations of the problem. If is assigned initial charge distribution, then problem consists in determining of particle trajectories and change of the charge distribution in the time, which corresponds to the obtained totality of trajectories (unsteady problem).

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In the second setting initial charge distribution is not assigned; problem consists in the determination of stationary charge distribution (not depending clearly on the time) and in the calculation of particle trajectories, at which is supported this stationary distribution (steady-state problem). Although the nonsteady-state problem is more interesting, its solution for the cases given below is not yet found. Therefore subsequently let us examine only stationary problems.

The radio engineering requirements, presented to the vacuum in the linear accelerator, are such, that the mean free path of particles exceeds the length of accelerator or compared with it. Therefore let us disregard/neglect particle scattering on the residual gas.

The general/common/total formulation of the problem of determining the self-congruent field disregarding by the effect of the collision of particles and by the effect of radiation/emission consists of the following. $n(x, y, z, P_x, P_y, P_z, t)$ — function of the distribution of density of particles in the six-dimensional phase space canonically conjugated/combined the variable/alternating r, P. In order to find a number of particles per unit of volume of the usual three-dimensional space x, y, z, let us integrate the function of the distribution of phase density according to all possible particle momenta at the given values of coordinates. Hence the density of space charge is equal to

$$\varrho(x, y, z, t) = e \int n(\mathbf{r}, \mathbf{P}, t)\, dP, \qquad (3.1)$$

where

$$dP = dP_x dP_y dP_z.$$

Current density is determined by the integral

$$\boldsymbol{\delta}(x, y, z, t) = e \int \mathbf{v}(\mathbf{r}, \mathbf{P}, t)\, n(\mathbf{r}, \mathbf{P}, t)\, dP. \qquad (3.2)$$

During the calculations of current density (3.2) let us disregard/neglect the transverse components of current and the scatter of particle speed, connected with the longitudinal vibrations. Then

$$\delta(x, y, z, t) = v_s \varrho(x, y, z, t).$$

Phase density in the space of the canonically conjugated/combined variable/alternating satisfies Liouville theorem dn/dt=0, or

$$\frac{\partial n}{\partial t} - \frac{\partial n}{\partial x} \cdot \frac{\partial x}{\partial t} - \ldots + \frac{\partial n}{\partial P_x} \cdot \frac{\partial P_x}{\partial t} + \ldots = 0.$$

Connecting to this equation the canonical equations of motion and equation for the scalar and vector potentials of field, we obtain the complete system of equations of self-congruent field [79]. Equations of motion can be written in the form

$$\mathbf{v} = \mathrm{grad}_P H, \quad \frac{d\mathbf{P}}{dt} = -\mathrm{grad}\, H.$$

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Index P with the gradient means that differentiation is conducted according to the components of generalized momentum; H - Hamiltonian of the particle

$$H = \mathcal{E} + eU.$$

Hamiltonian does not contain the explicitly vector potential of field, but, as it is possible to show [21],

$$\mathrm{grad}\, H = e\, \mathrm{grad}\, U - e\, [\mathbf{v} \cdot \mathrm{rot}\, \mathbf{A}] - e \frac{d\mathbf{A}}{dt}.$$

where U, A - sums of the corresponding potentials of applied fields and proper field of beam. This expression directly follows from formulas (2.1), (2.5). The system of equations of self-congruent field takes the form

$$\frac{\partial n}{\partial t}+\mathbf{v}\,\mathrm{grad}\,n-\mathrm{grad}\,H\,\mathrm{grad}_p n=0;$$

$$\mathbf{v}=\mathrm{grad}_p H; \qquad (3.3)$$

$$\frac{d\mathbf{P}}{dt}=-\mathrm{grad}\,H;$$

$$H=mc^2-eU;$$

$$\Delta U=-\frac{e}{\varepsilon_0}\int n\,dP;$$

$$\Delta\mathbf{A}=-\mu_0 e\int \mathbf{v}n\,dP.$$

We will consider that the particles move in the medium, which does not contain dielectrics,

$$\varepsilon_0=\frac{10^7}{4\pi c^2}\ \phi\,м;\ \mu_0=4\pi\cdot10^{-7}\ \overset{(1)}{гн\,м}. \qquad (3.4)$$

Key: (1). H/m.

System of equations (3.3) has countless solution set. Let us assume that are known the expressions for any first integrals of equations of motion

$$I_1(x,\ y,\ z,\ P_x,\ P_y,\ P_z,\ t)=\mathrm{const};$$
$$I_2(x,\ y,\ z,\ P_x,\ P_y,\ P_z,\ t)=\mathrm{const}.$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Then arbitrary function from these constants of motion will give solution for the function of the distribution of the phase density

$$n(x,\ y,\ z,\ P_x,\ P_y,\ P_z,\ t)=f(I_1,\ I_2,\ \ldots,), \qquad (3.5)$$

and, electromagnetic fields obtained, on the basis of this function of distribution of phase density will be self-consistent.

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Actually/really, first equation (3.3) is satisfied with the arbitrary function f

$$\frac{dn}{dt} = \frac{\partial f}{\partial I_1} \cdot \frac{dI_1}{dt} + \frac{\partial f}{\partial I_2} \cdot \frac{dI_2}{dt} + \ldots \equiv 0,$$

since, regarding constant of motion, dI/dt=0. Latter/last three equations (3.3) are satisfied inasmuch as along the condition constants of motion are obtained in the field of forces, assigned by these equations.

Expressions for constants of motion contain the unknown thus far potentials of electromagnetic field. Let us examine a special case, namely let us assume that the Hamiltonian

$$H = \mathscr{E} + e(U + U_1)$$

is constant of motion; here U - potential of the proper field of beam; U_1 - assigned potential of applied field. If external field in the course of time does not change, then it is possible to assume that and the potential of the proper field of beam also clearly on time does not depend and, consequently, also Hamiltonian as a whole does not depend clearly on time. According to expression (3.5), it can assume

$$n = f(H),$$

where f - arbitrarily assigned function of the distribution of phase density. Then equation for the scalar potential is reduced to the following:

$$\Delta U = -\frac{e}{\varepsilon_0} \int f(\mathscr{E} + eU + eU_1)\, dP,$$

but for the vector potential, according to expression (2.42), we will obtain

$$\mathbf{A} = \frac{\mathbf{v}}{c^2} V.$$

From the latter/last two equations are determined the potentials of self-congruent field with any assigned distribution function.

The equations of motion of particles in the strong-focusing linear accelerator taking into account the field of the space charge of beam are obtained in chapter 2 and take form (2.50). On the assumption that the amplitude of the longitudinal component of accelerating field does not depend on transverse coordinates and that the proper field of beam is absent, transverse and longitudinal coordinates were divided. However, when the proper field of beam is present, the problem in principle becomes complicated and in the general case is reduced to the determination of the function of the distribution of phase density in the six-dimensional phase space. However, preliminary estimations show that also taking into account space charge the problem can be simplified. In the process of phase stability the beam decomposes into the clusters. The aperture of channel and, therefore, the transverse sizes/dimensions of clusters is substantially less than $\beta\lambda$.

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Since the strong focusing virtually does not limit the selection of synchronous phase, the latter is selected so as to ensure the sufficiently wide capture region of particles on the phases. In this case the longitudinal length of each cluster is from 1/4 to 1.3 $\beta\lambda$. Thus, the longitudinal sizes/dimensions of clusters exceed transverse ones. In the clusters, elongated along the longitudinal axis, the transverse components of the proper field of beam weakly depend on the length of cluster. Let us show this based on the example of the evenly charged/loaded general ellipsoid. The approximation of clusters by the evenly charged/loaded ellipsoid was previ usly used in works [116.3] for the evaluation of the effect of the pushing apart Coulomb forces in the beam. r_x, r_y, l — semi-axis of ellipsoid, moreover the longitudinal semi-axis l is more than transverse semi-axes r_x, r_y. The potential of field within evenly charged/loaded general ellipsoid [80] is equal to

$$U = -\frac{\varrho}{2\varepsilon_0}\left[\frac{M}{2}(x^2+y^2) - N(x^2-y^2) + (1-M)z^2\right].$$

where

$$M = 1 - \frac{r_x r_y l}{2}\int_0^\infty \frac{ds}{(r_x^2+s)^{1/2}(r_y^2+s)^{1/2}(l^2+s)^{3/2}};$$

$$N = \frac{r_x r_y (r_x^2 - r_y^2)}{4}\int_0^\infty \frac{ds}{(r_x^2+s)^{3/2}(r_y^2+s)^{3/2}(l^2+s)^{1/2}}.$$

Fig. 31 shows the dependence of coefficients M, N on relation r_x/l. From the graphs/curves it is evident that coefficient M weakly depends on relation r_x/l and it is close to unity up to comparatively large values r_x/l. Coefficient N depends on relation r_x/l is even less susceptibly/critically. Up to $l=2r_x$ with the accuracy better than 5o/o coefficient N remain equal to value

$$N=\frac{1}{2}\frac{r_x-r_y}{r_x-r_y}.$$

Therefore, examining transverse vibrations, we can assume $l=\infty$. i.e., disregard the decomposition/decay of beam on the clusters. This gives the overestimation of Coulomb forces to 10-15o/o. Apparently, in actuality error is still less, since was not considered the effect of the metallic walls of channel; furthermore, Coulomb pushing apart greatest with the low energies and should be considered also the effect of the particles, which proved to be out of the separatrix. If we consider clusters infinitely long, then thereby we is disregarded by the longitudinal vibrations of particles and at our disposal remain only the equations of transverse vibrations. The task of determining the self-congruent field is reduced to the four-dimensional. In this examination all particles are assumed to be synchronous ones and it is not possible to determine the form of cluster from the complete six-dimensional task taking into account the dependence of the defocusing action of high-frequency field on the phase of particle.

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Let us note that the difference 1-M susceptibly/critically depends on the relationship/ratio of the transverse and longitudinal sizes/dimensions of cluster. During the evaluation of the effect of longitudinal Coulomb pushing apart on the process of phase stability it is important to consider both the length of cluster and its transverse sizes/dimensions, since the longitudinal component of the natural field of beam depends substantially on these sizes/dimensions. Therefore first let us examine space-charge effect on the beam focusing let us determine the transverse sizes/dimensions of beam disregarding by the decomposition/decay of beam to the clusters. In Chapter 4 we examine space-charge effect on the phase stability, being assigned by the transverse sizes/dimensions of beam.

Latter/last task can be brought to the two-dimensional, if we assume that the longitudinal component of Coulomb field of the bundle does not depend on transverse coordinates. In the case of ellipsoid this by itself the cylinder of finite length, the assumption indicated is fulfilled only approximately; however, with the satisfactory accuracy. A relative difference in the longitudinal components of field on the axis and on the periphery of the evenly charged/loaded cylinder of finite length, elongated along the longitudinal axis, does not exceed 40o/o on the edge of cylinder and rapidly it decreases with the departure/attendance from the edge. Dependence on a radius of the longitudinal component of the proper field of the evenly charged/loaded circular cylinder with length-diameter ratio, equal to three, it is shown in Fig. 4.2.

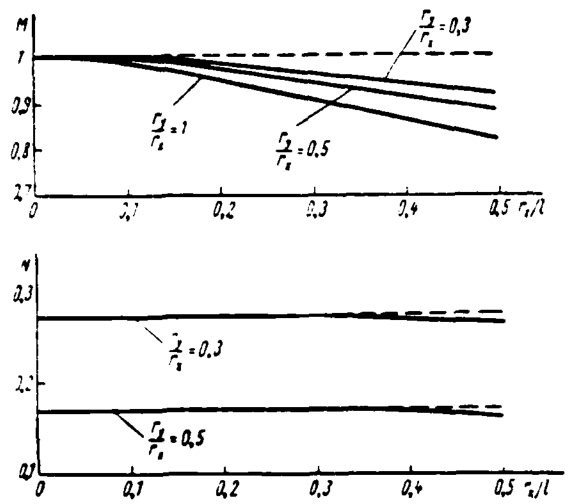

Fig. 3.1.

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S 3.2. Equations of transverse vibrations in the strong-focusing channel during the "microcanonical" distribution of phase density.

With the linear approximation of the focusing fields of the equation of transverse vibrations (2.50) they take the form

$$\frac{d^2x}{dt^2} = -\frac{ev_z}{m_0\gamma} G(z)\,x - \frac{e}{2m_0\gamma^3}\cdot\frac{\partial E_z}{\partial z}(z,t)\,x - \frac{e}{m_0\gamma^3}\cdot\frac{\partial U}{\partial x}(x,y,z);$$
$$\frac{d^2y}{dt^2} = \frac{ev_z}{m_0\gamma} G(z)\,y - \frac{e}{2m_0\gamma^3}\frac{\partial E_z}{\partial z}(z,t)\,y - \frac{e}{m_0\gamma^3}\frac{\partial U}{\partial y}(x,y,z). \quad (3.6)$$

According to the simplifying assumptions accepted, the longitudinal coordinates z of particles are connected with the current time t with relationship/ratio

$$z(t) = \int_{t_0}^{t} v_z(t)\,dt.$$

general/common/total for all particles so that any of these variable/alternating can be selected as the independent variable.

Equations (3.6) can be obtained from the canonical ones, if to the latter corresponds the Hamiltonian

$$H=\frac{p_x^2+p_y^2}{2m_0\gamma}+ev_sG(z)\,\frac{x^2-y^2}{2}+\frac{e}{2\gamma^2}\cdot\frac{\partial E_z}{\partial_z}(z)\,\frac{x^2-y^2}{2}+\frac{e}{\gamma^2}U(x,\,y,\,z).$$

In the linear approximation/approach to focusing fields $A_x = A_y = 0$ and transverse particle momenta are canonically conjugated/combined with the transverse Cartesian coordinates. As can easily be seen, Hamiltonian depends on time, so the gradient of focusing field - function of longitudinal coordinate. Therefore in the strong-focusing channel Hamiltonian is not constant of motion. Let us note that in this case the dependence of gradient $\frac{\partial E_z}{\partial z}$ on the independent variable is not so essential: because of the smallness of the factor of defocusing the gradient of accelerating field always can be with a sufficient accuracy replaced with the value, averaged on the period of the accelerating structure and, therefore, which does not depend on longitudinal coordinate. The fact that in the strong-focusing channel the Hamiltonian is not constant of motion, complicates task. To compose the first integral of equations of motion (3.6), which occurs with any distribution functions, is impossible. This difficulty can be overcome, after averaging equations (3.6) for the period of focusing field. The Hamiltonian of the averaged equations would not depend on time. However, "smooth" approximation/approach gives satisfactory accuracy only at the relatively small frequencies of transverse vibrations and does not befit, for example, during calculations of the matching networks. Therefore let us first solve task in the general view, without resorting to the averaging of the equations of motion.

For one important particular case of the distribution function it was possible to indicate the integrals of motion of unaveraged equation [81].

Let us pass for the convenience in equations (3.6) to the independent variable τ (2.52):

$$\frac{d^2x}{d\tau^2}+Q_x(\tau)\,x+\frac{eS^2}{\mathcal{E}_0\beta^2\gamma^3}\cdot\frac{\partial U}{\partial x}=0;\qquad \frac{d^2y}{d\tau^2}+Q_y(\tau)\,y+\frac{eS^2}{\varepsilon_0\beta^2\gamma^3}\cdot\frac{\partial U}{\partial y}=0. \tag{3.7}$$

Let us assume that there is such stationary distribution of phase density during which the components of the proper field of beam linearly depend on the corresponding transverse coordinates. In this case equations (3.7) will prove to be linear, with the divided variable/alternating it is possible to write the fundamental pairs of solutions in general form, without knowing the concrete/specific/actual dependence of the potential of proper field on the coordinates

$$\chi_x(\tau)=\sigma_x(\tau)\,e^{\pm i\psi_x(\tau)};\qquad \chi_y(\tau)=\sigma_y(\tau)\,e^{\pm i\psi_y(\tau)}. \tag{3.8}$$

In accordance with the condition for standardization (2.65)

$$\frac{d\psi_x}{d\tau}=\frac{1}{\sigma_x^2};\quad \frac{d\psi_y}{d\tau}=\frac{1}{\sigma_y^2}. \tag{3.9}$$

Real solutions will take the form

$$\begin{aligned} x(\tau) &= A_x\sigma_x(\tau)\cos[\psi_x(\tau)+\Theta_x];\\ y(\tau) &= A_y\sigma_y(\tau)\cos[\psi_y(\tau)+\Theta_y]. \end{aligned} \tag{3.10}$$

Values A_x, A_y, Θ_x, Θ_y depend on the initial values of variable/alternating x, y, $\frac{dx}{d\tau}$, $\frac{dy}{d\tau}$ and they remain constants in the process of motion, i.e., are constants of motion. According to expression (3.5), phase density is an arbitrary function of constants of motion. Let us assume that the phase density does not depend on the phases of transverse vibrations Θ_x, Θ_y:

$$n=f(A_x,\ A_y).$$

This means that at any point of channel and with any available amplitude of oscillations there are particles with all possible phases of transverse oscillations. Eliminating phases from solutions (3.10), we obtain expressions for two constants of motion

$$\begin{aligned} A_x^2 &= \left(\sigma_x\frac{dx}{d\tau}-\frac{d\sigma_x}{d\tau}x\right)^2+\left(\frac{x}{\sigma_x}\right)^2;\\ A_y^2 &= \left(\sigma_y\frac{dy}{d\tau}-\frac{d\sigma_y}{d\tau}y\right)^2+\left(\frac{y}{\sigma_y}\right)^2. \end{aligned} \tag{3.11}$$

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It is assumed that σ_x, σ_y— assigned functions of the independent variable. Let us introduce new constant of motion

$$F=A_x^2+A_y^2. \tag{3.12}$$

Since values A_x, A_y characterize the intensity of transverse oscillations in two mutually perpendicular planes, constant of motion F in a sense is equivalent to the integral of energy. Let us show that there is such dependence n=f(F), with which the equations of motion (3.7) actually/really prove to be linear and with the divided variable/alternating. This takes place when for all particles in the beam constant of motion F has one and the same value/significance of $F=F_0$. The dependence of phase density on constant of motion can be written in the form:

$$n=n_0\delta(F-F_0), \qquad (3.13)$$

where δ(x) - Dirac's delta function

$$\delta(x)=0 \text{ при}^{(1)}\ x\neq 0;\ \int_{-\infty}^{+\infty}\delta(x)\,dx=1.$$

Key: (1). with.

During this selection of the distribution function the representative points of all particles at the four-dimensional phase space $x,\ y,\ \dot{x},\ \dot{y}$ lie/rest on the three-dimensional surface of the hyper-ellipsoid

$$(\sigma_x\dot{x}-\dot{\sigma}_x x)^2+(\sigma_y\dot{y}-\dot{\sigma}_y y)^2+\left(\frac{x}{\sigma_x}\right)^2+\left(\frac{y}{\sigma_y}\right)^2=F_0. \qquad (3.14)$$

Let us design hyper-ellipsoid (3.14) on plane $x,\ \dot{x}$. Utilizing equations (2.310), we obtain

$$(\sigma_x\dot{x}-\dot{\sigma}_x x)^2+\left(\frac{x}{\sigma_x}\right)^2=F_0. \qquad (3.15)$$

Analogous equation is correct for the projection of hyper-ellipsoid on plane $y,\ \dot{y}$. According to expressions (3.12), (3.13),

$$A_x^2+A_y^2=F_0.$$

On ellipse (3.15) lie/rest the representative points of the particles, which move at plane XOZ; for such particles $y=\dot{y}=0$ and $A_y=0$. For remaining particles $A_x^2<F_0$ and their representative points fall inside ellipse (3.15). The representative points of all particles, which move in plane YOZ, are projected/designed into the origin of the coordinates of plane x, $\dot{x}$ since for these points $A_x=0$. Thus, if in the four-dimensional phase space the representative points lie/rest on the surface of hyper-ellipsoid, then in each two-dimensional phase space the representative points fill the volumes, included by ellipses, type (3.15).

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Area within ellipse (3.15), according to expressions (2.110), (2.114), is equal to πF_0. Consequently, F_0 - value of the given phase volume of beam for each of the phase planes x, $\dot{x}$ and y, $\dot{y}$. Since the variable/alternating in the equations of motion are divided, phase volumes must remain invariant on each plane x, p_x and y, p_y.

From equation (3.14) it follows that for any particle

$$\left(\frac{x}{\sigma_x}\right)^2+\left(\frac{y}{\sigma_y}\right)^2\leqslant F_0.$$

Out of the ellipse

$$\frac{x^2}{(\sigma_x\sqrt{F_0})^2}+\frac{y^2}{(\sigma_y(\sqrt{F_0})^2}=1 \qquad (3.16)$$

there are no particles. Particles with the maximum amplitudes in each transverse plane reach ellipse (3.16) at the moment/torque when

$$\sigma_x \dot{x} - \dot{\sigma}_x x = 0; \ \sigma_y \dot{y} - \dot{\sigma}_y y = 0.$$

Ellipse (3.16) covers beam section by plane XOY. The semi-axes of section are equal to

$$\begin{aligned} r_x(\tau) &= \sqrt{F_0}\,\sigma_x(\tau); \\ r_y(\tau) &= \sqrt{F_0}\,\sigma_y(\tau). \end{aligned} \qquad (3.17)$$

In this case it is assumed that functions $\sigma_x(\tau), \sigma_y(\tau)$ are selected so that on each phase plane ellipse (3.15) would coincide with the boundary of phase volume. Initial conditions for functions σ_x, σ_y are assigned by formulas (2.119). Are obtained the same relationships/ratios, that also in the single-particle theory, but the moduli/modules are fundamental of solutions $\sigma_x; \sigma_y$ thus far they are not determined.

Let us note that constant of motion F, determined by expression (3.12), led to the equality of areas on planes x, $\dot{x}$ and y, $\dot{y}$. The case when two-dimensional phase volumes are distinguished, can be considered, after introducing the factor, different from unity, with the afore-mentioned from the squares in sum (3.12). For simplification in the task let us assume that the beam at the entrance of channel possesses axial symmetry, so that its two-dimensional phase volumes are identical with respect to value.

The distribution of phase density (3.13) is analogous with the microcanonical distribution of the states of the isolated/insulated system according to energies [82]. Therefore distribution (3.13) let us name/call microcanonical.

According to expressions (3.1), (3.13), the density of space charge in each beam section is distributed according to the law

$$\varrho(x, y, z) = en_0 \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \delta(F - F_0)\, d\dot{x}\, d\dot{y}.$$

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Let us replace in the dual integral the variable/alternating integrations $\dot{x}$, y by the new variable/alternating α, Ω:

$$\sigma_x\dot{x} - \dot{\sigma}_x x = \alpha\cos\Omega;$$
$$\sigma_y\dot{y} - \dot{\sigma}_y y = \alpha\sin\Omega.$$

This replacement is equivalent to transition on plane $\sigma_x\dot{x}$, $\sigma_y\dot{y}$ from the Cartesian coordinates to the polar ones with the displacement of the origin of coordinates into point $\dot{\sigma}_x x$, $\dot{\sigma}_y y$. Function F(x, y, $\dot{x}$, $\dot{y}$, τ) is determined by expressions (3.12), (3.11) and in the new variable/alternating takes the form

$$F = \alpha^2 + \left(\frac{x}{\sigma_x}\right)^2 + \left(\frac{y}{\sigma_y}\right)^2,$$

and the jacobian of conversion is equal to [83]

$$\begin{vmatrix} \frac{\partial \dot{x}}{\partial \alpha} & \frac{\partial \dot{x}}{\partial \Omega} \\ \frac{\partial \dot{y}}{\partial \alpha} & \frac{\partial \dot{y}}{\partial \Omega} \end{vmatrix} = \frac{\alpha}{\sigma_x(\tau)\,\sigma_y(\tau)}.$$

Hence

$$d\dot{x}\,d\dot{y} = \frac{\alpha}{\sigma_x(\tau)\,\sigma_y(\tau)}\,d\alpha\,d\Omega,$$

so that

$$\varrho(x, y, \tau) = \frac{en_0}{\sigma_x\sigma_y}\int_0^{2\pi} d\Omega \int_0^{\infty} \delta\left(\alpha^2 + \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - F_0\right)\alpha\,d\alpha.$$

Let us introduce one additional replacement of variable/alternating $\alpha^2=u$ and will designate

$$u_0 = F_0 - \left(\frac{x}{\sigma_x}\right)^2 - \left(\frac{y}{\sigma_y}\right)^2.$$

Then

$$\varrho(x, y, \tau) = \frac{\pi e n_0}{\sigma_x\sigma_y}\int_0^{\infty} \delta(u - u_0)\,du.$$

If point x, y is located out of ellipse (3.16), then $u_0<0$ and $\rho(x, y, \tau)=0$. If point x, y lies/rests within ellipse (3.16), then $u_0>0$ we obtain

$$\varrho(x, y, \tau) = \frac{\pi e n_0}{\sigma_x(\tau)\,\sigma_y(\tau)}.$$

Thus, in each beam section the density of space charge does not depend on transverse coordinates. The microcanonical distribution of phase density leads to the uniform distribution of the charge density according to the beam section.

Let us assume I - maximum instantaneous current strength in each cluster of the accelerated beam.

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This value let us name/call peak beam current. Since we disregarded/neglected the decomposition/decay of beam to the clusters, then one should consider that we deal concerning the steady beam, in each section of which the current retains one and the same value/significance

$$I(z, t) = v_s \int\int \varrho(x, y, z, t)\, dx\, dy = \text{const},$$

equal to peak beam current. Then

$$\varrho(\tau) = \frac{I}{\pi v_s r_x r_y},$$

since each beam section - this is ellipse with the uniform distribution of charge. Taking into account expression (3.17)

$$n_0 = \frac{I}{\pi^2 e v_s F_0}.$$

Let us assume that a substantial change in the semi-axes of section occurs at the distances, which considerably exceed the values of semi-axes. This assumption is fulfilled well, since the transverse sizes/dimensions of beam usually are considerably lower than the period of focusing field. Coulomb field of particles rapidly decreases with the distance and for each given one τ the beam can be approximated by infinite elliptical cylinder with semi-axes $r_x(\tau), r_y(\tau)$. The scalar potential of the proper field of beam satisfies in this case the equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = -\frac{1}{\varepsilon_0}\varrho(\tau), \qquad (3.18)$$

where

$$\varrho(\tau)=\begin{cases} I/\pi v_s r_x r_y & \text{при (1)} \ \frac{x^2}{r_x^2}+\frac{y^2}{r_y^2}\leqslant 1 \\ 0 & \text{при (1)} \ \frac{x^2}{r_x^2}+\frac{y}{r_y^2}>1. \end{cases}$$

Key: (1). with.

Let us determine the potential of the evenly charged/loaded elliptical cylinder. The potential of field within the cylinder corresponds to the equation

$$\Delta U_i=-\frac{1}{\varepsilon_0}\varrho,$$

and out of the cylinder

$$\Delta U_e=0.$$

Function U is continuous on the surface of cylinder together with its first-order derivatives, and at infinity external potential behaves as the potential of charged/loaded straight line: $U_e\sim\ln r$, where r - the radius-vector of point in plane XOY. Let us switch over in plane XOY to elliptical coordinates [84]

$$x=f\operatorname{ch}\xi\cos\eta;\quad y=f\operatorname{sh}\xi\sin\eta. \qquad (3.19)$$

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The lines of the equal values of coordinates ξ are the ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where a, b - current semi-axes of the coordinate ellipse

$$a = f\,\mathrm{ch}\,\xi;\quad b = f\,\mathrm{sh}\,\xi.$$

Value f - half of focal distance, identical for all coordinate ellipses

$$f^2 = a^2 - b^2.$$

Let us select parameter f in such a way that at certain value/significance $\xi=\xi_0$ beam section would coincide with the coordinate curve

$$r_x = f\,\mathrm{ch}\,\xi_0;\quad r_y = f\,\mathrm{sh}\,\xi_0.$$

Hence

$$\frac{r_x - r_y}{r_x + r_y} = e^{-2\xi_0}. \qquad (3.20)$$

The lines of equal values η— this family of the hyperbolas

$$\frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} = 1,$$

where

$$a_1 = f\cos\eta;\quad b_1 = f\sin\eta,$$

moreover, the foci of coordinate hyperbolas and ellipses coincide.

The Laplacian of two-dimensional task in curvilinear coordinates takes form [84]

$$\Delta U = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial\xi}\left(\frac{h_2}{h_1}\cdot\frac{\partial U}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{h_1}{h_2}\cdot\frac{\partial U}{\partial\eta}\right)\right],$$

where h_1, h_2 - Lame's coefficients, equal to

$$h_1 = \sqrt{\left(\frac{\partial x}{\partial\xi}\right)^2 + \left(\frac{\partial y}{\partial\xi}\right)^2};$$

$$h_2 = \sqrt{\left(\frac{\partial x^2}{\partial\eta}\right)^2 + \left(\frac{\partial y}{\partial\eta}\right)^2}.$$

According to equalities (3.19), in elliptical curvilinear coordinates the Laplacian is led to the expression

$$\Delta U = \frac{1}{f^2(\mathrm{ch}^2\xi - \cos^2\eta)}\left(\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2}\right).$$

Hence we obtain equations for the potentials of internal and applied fields in the elliptical coordinates

$$\frac{\partial^2 U_i}{\partial \xi^2} + \frac{\partial^2 U_i}{\partial \eta^2} = -\frac{1}{\varepsilon_0}\varrho f^2(\mathrm{ch}^2\xi - \cos^2\eta),$$

$$\frac{\partial^2 U_e}{\partial \xi^2} + \frac{\partial^2 U_e}{\partial \eta^2} = 0.$$

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Particular solution of the nonhomogeneous equation

$$U_1(\xi, \eta) = -\frac{1}{8\varepsilon_0}\varrho f^2(\mathrm{ch}\,2\xi + \cos 2\eta).$$

General solution of the homogeneous equations consists of the sum of the terms

$$\mathrm{ch}\,n\xi\cos n\eta;\ \ \mathrm{ch}\,n\xi\sin n\eta;$$
$$\mathrm{sh}\,n\xi\cos n\eta;\ \ \mathrm{sh}\,n\xi\sin n\eta,$$

where n in view of the periodic dependence of potential on coordinate η— integers. from the considerations of symmetry the general solution must satisfy the conditions

$$U_0(\xi,\eta) = U_0(\xi, -\eta) = U_0(\xi, \pi-\eta).$$

Therefore the terms, which contain $\sin n\eta$, drop out, moreover in the sum remain only the even harmonics

$$U_0(\xi, \eta) = \sum_{m=0}^{\infty}(a_{2m}\,\mathrm{ch}\,2m\xi + b_{2m}\,\mathrm{sh}\,\xi)\cos 2m\eta.$$

Internal potential $U_i(x, y)$ must not have gaps in the foci of coordinate ellipses $\xi=0$; $\eta=0, \pi$. Hence it follows that for internal potential $b_{2m}=0$. Actually/really, let us examine one of the derivatives of the general solution

$$\frac{\partial U_0}{\partial x}=\frac{\partial U_0}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial U_0}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}.$$

From the transfer equations (3.19) we have

$$\frac{\partial \xi}{\partial x}=\frac{\operatorname{sh}\xi\cos\eta}{f(\operatorname{ch}^2\xi-\cos^2\eta)};\quad \frac{\partial \eta}{\partial x}=-\frac{\operatorname{ch}\xi\sin\eta}{f(\operatorname{ch}^2\xi-\cos^2\eta)}.$$

At the points of focus derivatives $\frac{\partial \xi}{\partial x}$, $\frac{\partial \eta}{\partial x}$ go to infinity. Derivative $\frac{\partial U_0}{\partial x}$ remains final. if in foci $\frac{\partial U_0}{\partial \xi}=\frac{\partial U_0}{\partial \eta}=0$, which occurs when $b_{2m}=0$. Thus, internal potential is determined by the expression

$$U_i(\xi,\eta)=-\frac{1}{\varepsilon_0}\varrho f^2(\operatorname{ch}2\xi+\cos 2\eta)+\sum_{m=0}^{\infty}a_{2m}\operatorname{ch}2m\xi\cos 2m\eta.$$

With $\xi\to\infty$ coordinate ellipses asymptotically approach the circles/circumferences, so that at infinity ξ - the logarithmic function of a radius - vector in plane XOY:

$$\xi=\ln\frac{a+b}{f}\to\ln\frac{2r}{f}.$$

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Hence it is apparent that the external potential satisfies boundary condition at infinity, if it consists of that damping with $\xi\to\infty$ the function and of the term, proportional ξ. Assuming/setting $b_{2m}=-a_{2m}$ for the external potential, we will obtain:

$$U_e(\xi,\eta)=b_0\xi+\sum_{m=1}^{\infty}b_{2m}e^{-2m\xi}\cos 2m\eta.$$

Further, from the boundary conditions on the surface of the elliptical cylinder

$$U_i(\xi_0,\eta)=U_e(\xi_0,\eta);\ \frac{\partial U_i}{\partial \xi}(\xi_0,\eta)=\frac{\partial U_e}{\partial \xi}(\xi_0,\eta)$$

it follows

$$a_2 = \frac{1}{8\varepsilon_0} \varrho f^2 e^{-2\xi_0};$$
$$a_{2m} = 0 \quad \text{при}^{(1)} \quad m > 1.$$

Key: (1). with.

Internal potential in the elliptical coordinates

$$U_i(\xi, \eta) = -\frac{1}{8\varepsilon_0} \varrho f^2 (\mathrm{ch}\, 2\xi + \cos 2\eta - e^{-2\xi_0} \mathrm{ch}\, 2\xi \cos 2\eta) - \mathrm{const}.$$

Let us return to the Cartesian coordinates. According to expressions (3.19),

$$x^2 + y^2 = \frac{1}{2} f^2 (\mathrm{ch}\, 2\xi - \cos 2\eta);$$
$$x^2 - y^2 = \frac{1}{2} f^2 (1 - \mathrm{ch}\, 2\xi \cos 2\eta).$$

Substituting latter/last equalities into the expression for the internal potential and taking into account equation (3.20), finally we obtain

$$U(x, y, \tau) = -\frac{1}{4\varepsilon_0} \varrho(\tau) \left[x^2 + y^2 - \frac{r_x - r_y}{r_x + r_y} (x^2 - y^2) \right] + \mathrm{const}. \quad (3.21)$$

From the expression for the potential of evenly charged/loaded elliptical cylinder (3.21) it is evident that the components of the proper field of **beam** $\frac{\partial U}{\partial x}$, $\frac{\partial U}{\partial y}$ — the linear functions of the corresponding transverse coordinates. By this is justified assumption about the linearity of equations of motion (3.7) and about the form of first integral (3.11), (3.12). Thus, during the stationary microcanonical distribution of phase density potential (3.21) describes self-congruent field.

Potential (3.21) consists of two components/terms/addends. First term corresponds to the field of beam with the axial symmetry, this field leads to the expansion of beam. Second term - to field with the quadrupole symmetry. The sign of second term always coincides with the sign of the potential of external quadrupole field, since in focusing on x lenses $r_x > r_y$. Second component of Coulomb potential does not shield, but amplifies external quadrupole field.

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During the conclusion/output of expression (3.21) it is assumed that the potential of the proper field of beam is completely determined by the charge of the accelerated particles; in other words, we is disregarded by the compensation for space charge by the secondary electrons, which are drawn in inside the positively charged/loaded beam. This is justified by the fact that in the ionic linear accelerators the slow secondary electrons are not seized into acceleration mode. In the strong-focusing channel the electron motion proves to be unstable, since due to the relatively small momentum of an electron fall in the unstable region it salted orders. In the channel appear the forces, which provoke the departure/attendance of electrons from potential well. Furthermore, in the proton accelerator-injectors the time of the establishment of compensated space charge [85] even without taking into account applied fields usually considerably exceeds the duration of pulse beam.

The redistribution of potential in the region, occupied with the space charge of beam, as is known, it limits the longitudinal velocity of particles with the assigned potential on the surface of beam [64]. This limits a maximally possible beam current. Limitation is determined by known law "three seconds". However, it has a value/significance only for the beam of particles low energies. For accelerated beams it is possible to disregard the effect of the redistribution of potential on the longitudinal velocity of particles. From expression (3.21) it is evident that the surface of the beam of elliptical cross section is not equipotential. Let us examine for simplicity the beam of round cross section $r_x = r_y = r_0$. Then

$$U = -\frac{1}{4\varepsilon_0}\varrho r^2 = -\frac{I}{4\pi\varepsilon_0 v}\left(\frac{r}{r_0}\right)^2.$$

Let on the surface of beam U=0. On the axis of bundle we have

$$\Delta U = \frac{I}{4\pi\varepsilon_0 v}.$$

Since $-eU=mv^2/2$, a relative change in the longitudinal velocity in the nonrelativistic approximation/approach will comprise $\Delta v/v=1/2\cdot\Delta U/U$, or

$$\frac{\Delta v}{v} = -\frac{e}{4\pi\varepsilon_0 m}\cdot\frac{I}{v^3}.$$

Let us produce numerical estimation, after accepting I=0.5 a; β=0.02. In this case of $\Delta v/v<0.2$o/o, which is negligibly small.

Further, during the conclusion/output of expression (3.21) they disregarded/neglected the effect of the metallic walls of channel on the potential distribution of proper field within the bundle. If bundle moves along the axis of continuous metal tube of round cross section, then when $r_x \neq r_y$ in potential (3.21) decreases the quadrupole component of field and appear nonlinear terms relative to x^2, y^2.

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As are shown the appropriate evaluations, the effect of metallic walls sufficiently little, if a radius of aperture opening/aperture somewhat exceeds the sizes/dimensions of the semi-axes of beam section. Thus, with $r_x = 7$ mm, $r_y = 3$ mm and a radius of channel 10 mm correction to linear component of the strength of field is less than 5o/o, but nonlinear component does not exceed 2o/o. The nearer the beam section to the circular, the less the correction. It is obvious that the circular metal tube does not affect potential distribution within the circular bundle whose axis coincides with the axis of duct.

Let us return to the equations of motion (3.7). After substituting in these equations derivatives of potential (3.21), we will obtain

$$\frac{d^2x}{d\tau^2} + Q_x(\tau)\,x - \frac{eIS^2}{2\pi m_0 c^3\beta^3\gamma^3\varepsilon_0}\cdot\frac{2x}{r_x(r_x+r_y)} = 0;$$
$$\frac{d^2y}{d\tau^2} + Q_y(\tau)\,y - \frac{eIS^2}{2\pi m_0 c^3\beta^3\gamma^3\varepsilon_0}\cdot\frac{2y}{r_y(r_x+r_y)} = 0.$$

The combination of the constants

$$I_0 = 4\pi\varepsilon_0\frac{m_0c^3}{e} \qquad (3.22)$$

has a dimensionality of current. For protons $I_0=3.14\cdot10^7$a. For the decrease of recordings let us introduce into the equations the parameter, which has the dimensionality of length,

$$r_a = S\sqrt{\frac{2I}{\beta^3\gamma^3 I_0}}. \qquad (3.23)$$

Then equations of motion taking into account the proper field of bundle are reduced to the form

$$\begin{aligned} \frac{d^2x}{d\tau^2} + \left[Q_x(\tau) - \frac{2r_a^2}{r_x(r_x+r_y)}\right]x = 0; \\ \frac{d^2y}{d\tau^2} + \left[Q_y(\tau) - \frac{2r_a^2}{r_y(r_x+r_y)}\right]y = 0. \end{aligned} \qquad (3.24)$$

If beam current is negligible, then it is possible to assume $r_a = 0$ and equations (3.24) are reduced to the equations of Mathieu-Hill type with assigned periodic coefficients Q_x, Q_y, i.e. to the equations of single-particle theory. For these equations by known methods can be calculated the functions of Floquet, the completely describing fluctuations of particles in the matched and unmatched beams. But if Coulomb terms are not small, then equations (3.24) directly are not solved - they contain unknown thus far functions $r_x(\tau)$, $r_y(\tau)$. Basic task consists of the calculation envelope of particles r_x, r_y. Equations for envelope taking into account Coulomb terms will be derived below. Under the specified initial conditions for equation for the envelopes have the periodic solutions with the period of focusing field.

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In this case equations (3.24) for the individual trajectories and when $r_a \neq 0$ prove to be equations of Mathieu-Hill type, and the functions

$$\varrho_x(\tau)=\frac{r_x(\tau)}{\sqrt{F_0}};\quad \varrho_y(\tau)=\frac{r_y(\tau)}{\sqrt{F_0}} \qquad (3.25)$$

are the moduli/modules of the corresponding functions of Floquet. Thus, for the matched beam it is possible to determine Floquet's functions taking into account Coulomb pushing apart. Under the mismatched initial conditions for enveloping equations individual trajectories (3.24) do not have periodic coefficients and for them not at all exists Floquet's functions. Therefore in contrast to the single-particle theory when $r_a \neq 0$ unmatched beams it is not possible to describe with the aid of Floquet functions and for them it is necessary each time to search for new solutions. Subsequently we will obtain the estimations, which make it possible to establish/install, in what cases it is possible to assume/set $r_a = 0$.

S 3.3. Envelope of particles. Frequency of incoherent.

Equations for the envelopes can be comprised analogously with equations (2.69). Let us substitute in the equations of motion (3.24) complex solutions (3.8). Taking into account relationships/ratios (3.9) and (3.17), we obtain

$$\frac{d^2r_x}{d\tau^2} - Q_x(\tau)\,r_x - \frac{F_0^2}{r_x^3} - \frac{2r_a^2}{r_x+r_y} = 0;$$
$$\frac{d^2r_y}{d\tau^2} - Q_y(\tau)\,r_y - \frac{F_0^2}{r_y^3} - \frac{2r_a^2}{r_x-r_y} = 0. \tag{3.26}$$

Initial conditions for the envelopes are determined by expressions (2.121).

Envelope of particles in the strong-focusing channel satisfy system of two nonlinear second order equations with the periodic coefficients, moreover variable/alternating in the equations are not divided. Equations (3.26) do not have the tabulated solutions and must be solved numerically under the assigned initial conditions. In the general case of solving the system of equations (3.26) they prove to be noncyclic that it corresponds to the envelopes of unmatched beams. Periodic solutions for the envelopes, that correspond to matched beam, determine from periodicity condition (with the period of functions Q_x, Q_y). Usually by known methods calculate the moduli/modules of Floquet's functions when $r_a = 0$, and then of the special computational program they find periodic solutions for r_x, r_y, gradually increasing parameter r_a to the given value. For the numerical integration of equation (3.26) it is convenient to convert, after passing directly to the moduli/modules of the standardized/normalized complex solutions by formulas (3.17).

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Then in the equations remains, besides assigned functions Q_x, Q_y, one arbitrary parameter

$$\frac{d^2\sigma_x}{d\tau^2} - Q_x(\tau)\,\sigma_x - \frac{1}{\sigma_x^3} - \frac{2r_a^2}{F_0}\cdot\frac{1}{\sigma_x+\sigma_y} = 0;$$
$$\frac{d^2\sigma_y}{d\tau^2} - Q_y(\tau)\,\sigma_y - \frac{1}{\sigma_y^3} - \frac{2r_a^2}{F_0}\cdot\frac{1}{\sigma_x+\sigma_y} = 0. \qquad (3.27)$$

The periodic solutions of equations (3.27) $\varrho_x(\tau)$, $\varrho_y(\tau)$ — the moduli/modules of Floquet functions of equations (3.24), that correspond to the specific value/significance of parameter $2r_a^2/F_0$. The definitions, given above for the instantaneous and medium frequencies of transverse vibrations (2.101), (2.104), can be now generalized to the case of matched beams with the large space charge. Phase change of Floquet's function in the matched beam is as before determined by formula (2.105)

$$\mu = \int_0^1 \frac{d\tau}{\varrho_x^2} = \int_0^1 \frac{d\tau}{\varrho_y^2}. \qquad (3.28)$$

To the numerical solution in the computers of equations (3.26) or (3.27) one should resort when previously it cannot be assumed that the hardnesses of quadrupole lenses are sufficiently small. But if when $r_a = 0$ phase change of transverse vibrations in the period of focusing field is small, then equations for the envelopes can be solved in the smooth approximation/approach. Let when $r_a = 0$ $\mu=\mu_0$. Let us assume $\mu_0 \ll 2\pi$. We will search for solutions for the envelopes in the form

$$r_x(\tau)=R_x(\tau)[1+q_x(\tau)];$$
$$r_y(\tau)=R_y(\tau)[1-q_y(\tau)]. \qquad (3.29)$$

Functions $q_x(\tau)$, $q_y(\tau)$ let us determine by equation (2.224) and conditions (2.226), (2.227). As was shown in S 2.7, from the condition $\mu_0 << 2\pi$ follows $q_x \ll 1$, $q_y \ll 1$. Values R_x, R_y - the average/mean values of envelopes in each period of focusing field. Since q_x, q_y are small,

$$\frac{1}{r_x^3} \approx \frac{1}{R^3}(1-3q_x).$$

$$\frac{1}{r_x-r_y} \approx \frac{1}{R_x+R_y} - \frac{R_x}{(R_x+R_y)^2}q_x - \frac{R_y}{(R_x+R_y)^2}q_y.$$

After substituting solutions (3.29) in equations (3.26), we will be restricted to linear approximation/approach on q_x, q_y. Averaging equations for the period of focusing field and taking into account equality (2.229), we obtain

$$\frac{d^2R_x}{d\tau^2}+\mu_0^2R_x-\frac{F_0^2}{R_x^3}-\frac{2r_a^2}{R_x+R_y}=0;$$
$$\frac{d^2R_y}{d\tau^2}+\mu_0^2R_y-\frac{F_0^2}{R_y^3}-\frac{2r_a^2}{R_x+R_y}=0. \qquad (3.30)$$

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Regarding, functions q_x, q_y from the beam current and on its phase volume do not depend. Thus, in the smooth approximation/approach the dependence of envelopes on the current and the phase volume is completely determined by slow components R_x, R_y. Equations (3.30) contain only constant coefficients, which in principle facilitates the calculations envelope of particles.

Let us examine first the focusing of matched beams in the smooth approximation/approach. Equations (3.30) have particular solution $R_x=\text{const};\ R_y=\text{const}$. In this case $\frac{d^2R_x}{d\tau^2}=\frac{d^2R_y}{d\tau^2}=0$, so that two coordinates of the state of equilibrium R_x, R_y are determined by the algebraic equations

$$\begin{aligned}\mu_0^2R_x-\frac{F_0^2}{R_x^3}-\frac{2r_a^2}{R_x+R_y}&=0;\\ \mu_0^2R_y-\frac{F_0^2}{R_y^3}-\frac{2r_a^2}{R_x+R_y}&=0.\end{aligned}\qquad(3.31)$$

Since equations (3.31) are symmetrical relative to the adjustment/exchange of indices, must exist solution $R_x=R_y=R_c$. System of equations (3.31) is reduced to one biquadratic equation

$$R_c^4-\frac{r_a^2}{\mu_0^2}R_c^2-\frac{F_0^2}{\mu_0^2}=0,$$

whence

$$R_c=\sqrt{\frac{r_a^2+\sqrt{r_a^4+4\mu_0^2F_0^2}}{2\mu_0^2}}.\qquad(3.32)$$

Envelope of particles, which correspond to this solution, are periodical with the period $\Delta\tau=1$

$$\begin{aligned}r_x(\tau)&=R_c[1+q_x(\tau)];\\ r_y(\tau)&=R_c[1+q_y(\tau)].\end{aligned}\qquad(3.29a)$$

The mean radius of matched beam R_c depends on the full current of beam, proportional, according to expression (3.23), parameter r_a^2, and from the value of the projection of the phase volume of beam on each of the phase planes, proportional F_0. From expression (3.32) it follows that with

$$r_a^2 \ll 2\mu_0 F_0, \qquad (3.33)$$

the mean radius of beam on current does not depend

$$R_c = R_c^0 = \sqrt{\frac{F_0}{\mu_0}},$$

where R_c^0 — the mean radius of matched beam with the negligible intensity.

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In other limiting case

$$r_a^2 \gg 2\mu_0 F_0$$

the mean radius equals

$$R_c = \frac{r_a}{\mu_0}$$

and does not depend on the phase volume of beam. With satisfaction of condition (3.33) it is possible to use the equations, which do not consider Coulomb interaction of particles. As the criterion of space-charge effect on the transverse size/dimension of matched beam is convenient value

$$h = \frac{r_a^2}{2\mu_0 F_0}, \qquad (3.34)$$

which let us name/call the Coulomb parameter of beam. Case (3.33) is reduced then to condition h<<1, and second limiting case - to condition h>>1. Expression for the mean radius of matched beam (3.32) can be simplified

$$R_c = R_c^0 \sqrt{h - \sqrt{1 - h^2}}. \qquad (3.35)$$

Thus, in the smooth approximation/approach the expansion of beam in the focusing channel is determined by the only parameter, which represents the dimensionless combination of the basic parameters of beam and channel. The Coulomb parameter of beam can be represented, according to expressions (2.106), (2.113), (3.23), in the form

$$h = \frac{c}{\beta\gamma^3 I_0} \cdot \frac{1}{\Omega_r^0} \cdot \frac{I}{V_n}. \qquad (3.36)$$

Here I_0 - characteristic for each type of the accelerated particles strength of current (3.22); Ω_r^0 — the medium frequency of transverse vibrations in the absence of space charge; I - complete peak beam current; V_n — the value of the projection of four-dimensional phase volume on one of phase planes (2.2). The ratio of the full current of beam in the peak to two-dimensional transverse phase volume V_n let us name/call the phase current density of beam. The Coulomb parameter of beam is proportional to phase current density.

From an increase in the energy of particles the Coulomb parameter falls. Therefore the space charge of beam most of all affects particle focusing in the initial part of the accelerator.

In the smooth approximation/approach to calculate the frequency of single particles in the matched beam is simple. The modulus/module of Floquet's function is determined by expressions (3.25), (3.29a), (3.35):

$$\varrho_x(\tau) = \frac{1+q_x(\tau)}{\sqrt{\mu_0}} \sqrt{h - \sqrt{1-h^2}}. \qquad (3.37)$$

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Substituting Floquet's modulus/module into equality (2.101), we have

$$\omega_r(\tau) = \omega_r^0(\tau)\left(\sqrt{1-h^2} - h\right), \qquad (3.38)$$

where ω_r^0 — the instantaneous value/significance of the frequency of transverse vibrations in the absence of space charge. It is averaged frequency on the period of the focusing field

$$\mu = \mu_0\left(\sqrt{1+h^2} - h\right). \qquad (3.39)$$

With an increase in the phase current density the frequency of transverse vibrations decreases. Consequently, smooth approximation/approach is improved for the beams with the large intensity. If particle motion in the absence of space charge is stable, then $|\cos\mu_0| < 1$ and μ_0 is real. Then frequency μ remains real at any values of Coulomb parameter. Thus, beam does not lose lateral stability with as the conveniently high currents. However, with the time of ripening of phase current density increase dimensions of beam in the assigned focusing fields; therefore to realize lateral stability with $h \to \infty$ in the principle is possible only with the infinite aperture.

For evaluating the limits of the applicability of the smooth approximation/approach of equation (3.27) they integrated numerically with the aid of electronic computer M-20. For each set of the parameters μ_0, $\frac{2r_a^2}{F_0}$ were determined the periodic solutions with the period of the focusing field: $\sigma_x = \varrho_x(\tau); \sigma_y = \varrho_y(\tau)$. Then they calculated the medium frequency of transverse vibrations according to formula (3.28). Fig. 3.2 gives the dependence of the relation of medium frequencies μ/μ_0 on value 1 - cos μ_0 for the symmetrical periods of the type FODO with a relative length of lenses of d=D/S, equal to d=1/20 and d=1/2.

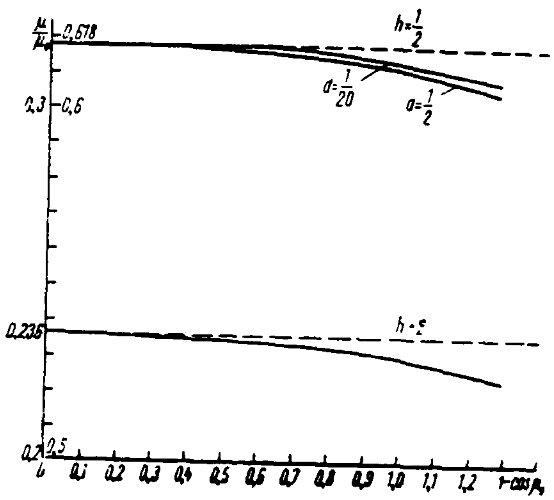

Fig. 3.2.

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Graphs/curves are given for two values of the Coulomb parameter of beam. On the vertical axis the right scale pertains to $h=1\,2$. left - to h=2. In the nasty approximation/approach the ratio μ/μ_0 is uniquely determined by the given value of the Coulomb parameter and on cos μ_0 does not depend. From the graphs/curves given in Fig. 3.2 it is evident that the medium frequency is calculated in the smooth approximation/approach with the high accuracy. A relative error in the calculation μ with cos $\mu_0 \geqslant -0.2$ does not exceed 5o/o.

Fig. 3.3 and 3.4 give the maximum $\varrho_x(0)$ and minimum $\varrho_y(0)$ values of Floquet's modulus/module in the channel FODO with d=1/2. Graphs/curves are constructed for three values of the Coulomb parameter h, equal to 0; 1/2; 2. Dotted line showed the outer limits of modulus/module, calculated in the smooth approximation/approach according to formula (3.37). Function $q_x(\tau) = -q_y(\tau)$ corresponding to this case is given in Fig. 2.20.

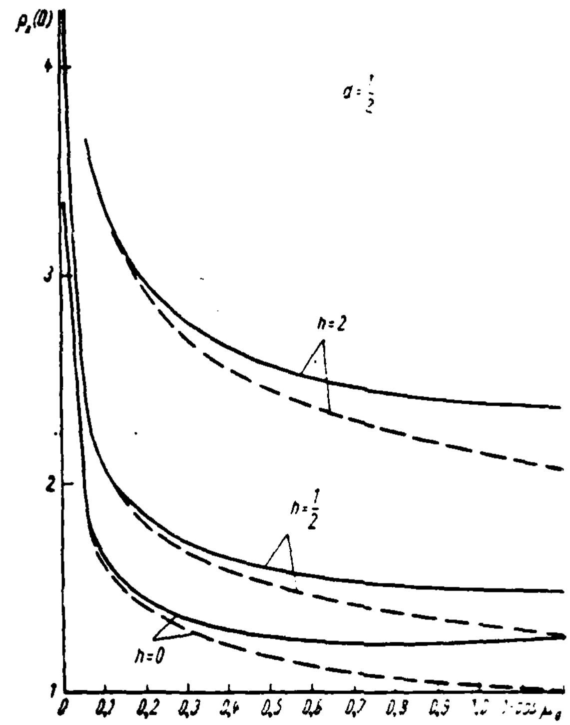

Fig. 3.3.

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From the graphs/curves it follows that smooth approximation/approach gives the error in the determination of the matched values of envelopes, which does not exceed by 10o/o to value/significance of cos μ_0=0.3. This value/significance corresponds to the approximately/exemplarily five periods of focusing field for one period of transverse vibrations (in the absence of noticeable space charge). With an increase of the Coulomb parameter relative error in the determination of the values of Floquet's modulus/module falls. The graphs/curves, analogous to those constructed in Fig. 3.3 and 3.4, occur also for the channel with short lenses (d=1/20). With cos $\mu_0 \geqslant 0.3$ an error in the smooth approximation/approach is virtually identical for the channels to the long and short lenses. At smaller values of cos μ_0 the error in the channel with the short lenses increases more rapidly than in the channel with the long lenses.

If at the entrance of channel Floquet's modulus/module does not have an extremum, then the definition of the matched initial conditions requires the calculations of both moduli/modules and derivatives. Fig. 3.5 and 3.6 give the results of the numerical calculations by the envelope of matched beam with the asymmetric period of the type FOD with two clearances of relative length 0.1 and 0.25 and with two lenses with a length of $d = 0.325$ each (model of the period of focusing field in the linear accelerator of protons I-2).

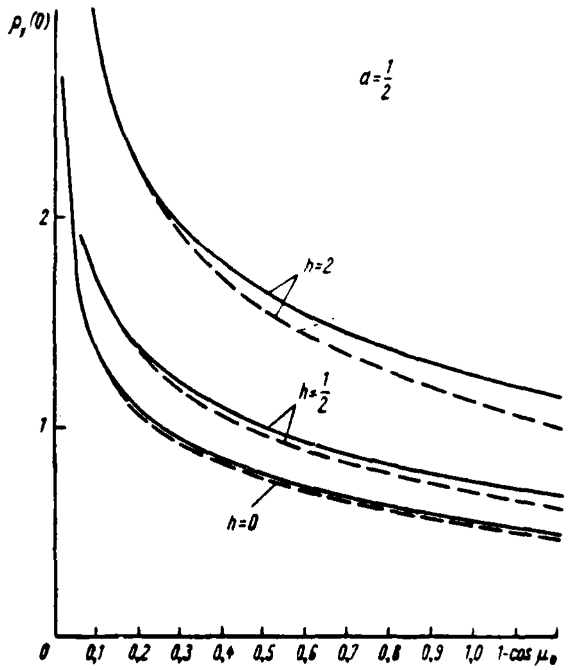
$\rho_1(0)$
$a=\frac{1}{2}$
2
1
0
$h=2$
$h=\frac{1}{2}$
$h=0$
0,1
0,2
0,3
0,4
0,5
0,6
0,7
0,8
0,9
1,0
$1-\cos\mu_0$

Fig. 3.4.

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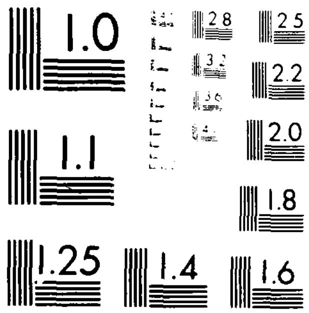

MICROCOPY RESOLUTION TEST CHART

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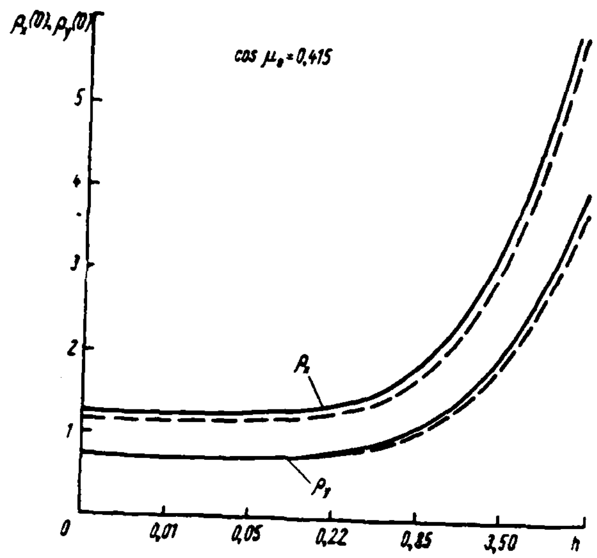

Fig. 3.5

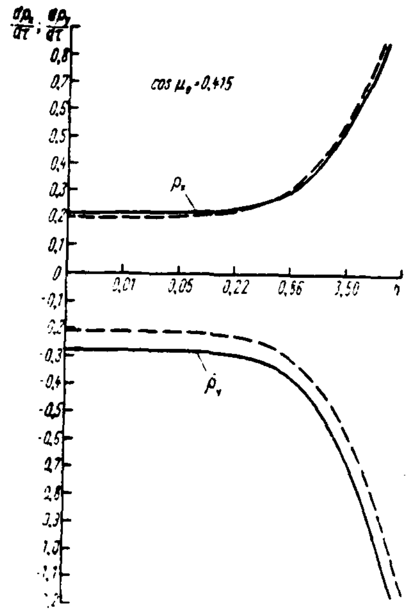

Fig. 3.6.

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Along the axis of abscissas are deposited/postponed the values of the Coulomb parameter on the logarithmic scale. Calculations are carried out to the high values of the Coulomb parameter and in Fig. 3.5 and 3.6 are given for one value/significance of medium frequency cos μ_0=0.415. Dotted line corresponds to the values, calculated in smooth approximation/approach (3.37), moreover in this case

$$\dot{Q}_x(0) = -\dot{Q}_y(0) = \frac{\dot{q}_{0x}}{\sqrt{\mu_0}}\sqrt{h+\sqrt{1-h^2}}$$

The approximate curves follow precise ones in entire range of change of the Coulomb parameter. The error in the determination of Floquet's modulus/module does not exceed 10o/o, while the error in the determination of derivatives of modulus/module composes 20o/o at the low values of h and decreases with an increase in the Coulomb parameter. Taking into account that the experimental determination of the absolute value of the phase volume of beam hardly can be produced on the real beam with the higher accuracy, the errors of the approximate computations should be considered completely satisfactory.

Thus, for calculating the matched initial conditions they must be assigned: to function $Q_x(\tau)$, $Q_y(\tau)$ of the chosen channel: the two-dimensional phase volume of beam V_n: the full current of beam in peak I. According to these data it is possible to find reduced volume (2.111) and characteristic Coulomb length (3.23) on the entrance of channel, which makes it possible to write equations for the envelopes (3.26). With the aid of electronic computer is determined the only periodic solution of these equations with the period $\Delta\tau=1$. If solutions are found, then this determines the parameters of matched beam at the entrance of channel $r_x(0)$, $r_y(0)$, $\frac{dr_x}{d\tau}(0)$, $\frac{dr_y}{d\tau}(0)$. Instead of two latter/last derivatives it is possible to determine the inclinations/slopes of envelope in the radians

$$\frac{dr_x}{dz}=\frac{1}{S}\cdot\frac{dr_x}{d\tau};\ \frac{dr_y}{dz}=\frac{1}{S}\cdot\frac{dr_y}{d\tau}.$$

As it was shown above, in many practical cases it is possible to be restricted to the smooth approximation/approach, which makes it possible not to resort to the numerical solution of equations (3.26) or (3.27). In the smooth approximation/approach the initial parameters of matched beam take the form

$$r_x(0)=\sqrt{\frac{F_0}{\mu_0}}\,[1+q_x(0)]\,\sqrt{h+\sqrt{1+h^2}};$$
$$r_y(0)=\sqrt{\frac{F_0}{\mu_0}}\,[1-q_y(0)]\,\sqrt{h+\sqrt{1-h^2}};$$
$$\frac{dr_x}{d\tau}(0)=\sqrt{\frac{F_0}{\mu_0}}\cdot\frac{dq_x}{d\tau}(0)\,\sqrt{h-\sqrt{1+h^2}};$$
$$\frac{dr_y}{d\tau}(0)=\sqrt{\frac{F_0}{\mu_0}}\cdot\frac{dq_y}{d\tau}(0)\,\sqrt{h-\sqrt{1+h^2}}.$$

Functions $q_x(\tau)$, $q_y(\tau)$, determined by equation (2.224) and conditions (2.226), (2.227), make it possible to find the medium frequency of transverse vibrations μ_0 (2.229) and the Coulomb parameter of beam on the entrance of channel (3.34).

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In the nasty approximation/approach the relation of the semi-axes of the matched section and the relation of the inclinations/slopes of envelopes are determined only by the parameters of the focusing channel and do not depend on phase volume and beam current. However, the absolute values of semi-axes and inclinations/slopes depend substantially on the phase volume of beam and phase current density.

With a change in these parameters it is necessary to again select matching conditions.

Let us examine as an example the entrance of the linear accelerator of protons I-100 (see S 2.9). Since the period of focusing field is symmetrical and begins from the middle of lens, matched beam must have at the entrance of accelerator a crossover. If phase current density is negligible, then h=0 and at the entrance we have

$$r_x^0 = \sqrt{\frac{F_0}{\mu_0}}\,[1 - q_x(0)];\ r_y^0 = \sqrt{\frac{F_0}{\mu_0}}\,[1 - q_y(0)].$$

We will use the fact that Floquet's modulus/module reaches at the entrance of the channel of outer limit $(\varepsilon=0)$ and it is converted formula for r_x^0, r_y^0. According to expressions (2.156), (2.234),

$$\nu_x = \frac{\mu_0}{(1+q_x)^2}.$$

Hence

$$r_x^0 = \sqrt{\frac{F_0}{\nu_\Phi}};\ \ r_y^0 = \sqrt{\frac{F_0}{\nu_\lambda}}. \qquad (3.40)$$

Relationships/ratios (3.40) are valid in the general case, since when $\varepsilon=0$ the relation of the semi-axes of the ellipse, which limits the phase volume of matched beam, is equal to ν. $V_n = 0.1$ cm•mrad. In the case of S=2βλ in question and in accordance with formula (2.113) F_0=0.04 cm². The instantaneous values of frequencies calculate from the matrix/die period and with cos μ_0=0.6 they are equal to

$\nu_{\phi} = 0{,}35$; $\nu_x = 1.9$. Hence $r_x^0 = 0{,}35$ cm; $r_y^0 = 0{,}15$ cm. Let us assume now that the peak beam current is led to 400 mA with phase volume $V_n = 0{,}2$ cm x mrad. Then we have F_e=0.08 cm² and h=0.68; this gives r_x $0{,}66$ cm, $r_y = 0{,}28$ cm. The relation of the semi-axes of section is retained.

If phase current density is negligible, then the mean radius of unmatched beam, as it was shown in Chapter 2, it oscillates with the doubled medium frequency of transverse vibrations of single particles. From the frequency of local maximums it is possible to judge the medium frequency of transverse vibrations of particles in the beam. The maximum sizes of unmatched beam are determined by its principal maximums and they always exceed the maximum size of matched beam with the same phase volume.

At the essential phase current density the picture of the behavior of the envelope of unmatched beam considerably becomes complicated. The repetition frequency of principal maximums increases, but the value of principal maximums which was kept approximately constant with h=0 (see Fig. 2.14), begins to oscillate with the relatively low frequency, much less than Ω_r^0.

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The integral relationship/ratio between the medium frequency of the oscillations of particles Ω, and the repetition frequency of the principal maximums of envelope is disrupted, so that from the frequency of local maximums it cannot be more judged the medium frequency of the fluctuations of particles within the beam. As in the case of $h=0$, with $h \neq 0$ the maximum size of unmatched beam in a rather long channel always exceeds the size/dimension of matched beam. With the decrease of phase current density the frequency of principal maximums is decreased and within the limit it vanishes. The envelope of unmatched beam with the high phase current density is schematically depicted in Fig. 3.7. With the virtually attainable at present values of phase current density the oscillations of principal maximums can be revealed only in the very long channels. Thus, with $h=1$ (which is close to the Coulomb parameter of the beam which can be achieved/reached in the contemporary proton accelerators) the period of oscillations of principal maximums is approximately 300 periods of focusing field. In the linear accelerators the Coulomb parameter decreases with an increase in the energy of particles, so that also this period gradually grows/rises. Thus, if are assigned initial conditions, then the maximum size of unmatched beam in the channel virtually completely is determined by the first principal maximum.

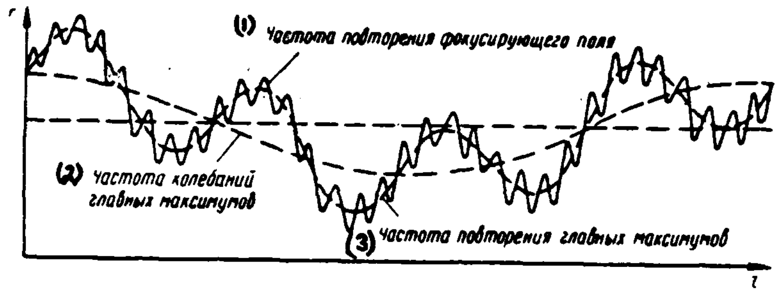

Fig. 3.7. Key: (1). Repetition frequency of focusing field. (2). Frequency of oscillations of principal maximums. (3). Repetition frequency of principal maximums.

§ 3.4. Focusing of matched beams during different stationary distributions of phase density.

All given relationships/ratios, which are determining the effect of Coulomb pushing apart on transverse vibrations of particles, are obtained on the assumption that occurs the stationary microcanonical distribution of phase density in four-dimensional phase space (3.13). In this case each beam section in the strong-focusing channel is limited by ellipse with uniform density distribution of space charge according to the section and equations of motion prove to be linear with the divided variable/alternating.

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During the microcanonical distribution the representative points of particles lie/rest only on the surface of four-dimensional hyper-ellipsoid; the four-dimensional phase volume of such a beam is equal to zero. Real beams possess different from zero four-dimensional phase volumes. In connection with this it is important to explain, as are changed the obtained relationships/ratios, if we switch over to other stationary distributions of phase density.

In § 3.3 it is shown that smooth approximation/approach gives completely satisfactory accuracy in the entire virtually interesting part of the first stability region. On the other hand, smooth approximation/approach makes it possible to pass from the equations of motion whose coefficients clearly depend on time, to the autonomous equations, i.e., to the equations with the constant coefficients. If we average equations of motion (3.7) along the period of focusing field, then Hamiltonian corresponding to these averaged equations proves to be in the matched beams constant of motion. Then it is possible to determine the self-consistent solutions for the averaged motion, on the basis of the fact that the function of the distribution of phase density depends on constants of motion of the averaged equations. Essential is the fact that the actual trajectories of particles are connected with the solutions of the averaged equations through the periodic coefficients whose value depends not on space charge, nor on the distribution of phase density, but it depends only on focusing fields themselves. Therefore constant of motion of averaged equations remains in the approximation/approach accepted and the integral of the initial unaveraged equations. In the smooth approximation/approach it is possible to present task to the end/lead and during some other stationary distributions, different from the microcanonical. Subsequently let us examine the following distributions.

1. Distribution, analogous to distribution of Fermi for degenerate electron gas [82]:

$$\begin{aligned} n = n_0 &= \text{const} \quad \overset{(1)}{\text{при}} \quad H \leqslant H_0; \\ n &= 0 \quad \overset{(1)}{\text{при}} \quad H > H_0. \end{aligned} \qquad (3.41)$$

Key: (1). with.

Distribution of this type we utilize also in Chapter 4 in the examination of the longitudinal vibrations of particles.

2. Distribution, analogous to canonical [82]:

$$n = n_0 e^{-\frac{H}{\theta}}, \qquad (3.42)$$

and leading to Maxwellian particle distribution according to thermal velocities.

In the preceding/previous paragraph smooth approximation/approach was used for the averaging of equations (3.26) whose form already corresponded to the concrete/specific/actual function of particle distribution in the self-congruent field.

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We now utilize smooth approximation/approach directly for the averaging of general/common/total equations of motion (3.7), making thus far no assumptions about the form of the function of distribution. Transition to the autonomous equations can be completed only for the matched beams, since with the unmatched beams in the Coulomb terms of equation remains explicit dependence on the time.

Thus, let us assume that the medium frequency of transverse vibrations of particles is much lower than repetition frequency of the focusing structure. Let us decompose derivatives of potential, entering equations (3.7), the power series for the low values of the rapid components of trajectory we will be restricted to linear terms. According to expression (2.222),

$$\begin{aligned} x(\tau) &= X(\tau) + q_x(\tau) X(\tau); \\ y(\tau) &= Y(\tau) + q_y(\tau) Y(\tau). \end{aligned} \qquad (3.43)$$

For derivative $\frac{\partial U}{\partial x}$ we have

$$\frac{\partial U}{\partial x}(x,\ y,\ \tau)=\frac{\partial U}{\partial x}(X,\ Y,\ \tau)-\frac{\partial^2 U}{\partial x^2}(X,\ Y,\ \tau)\,q_xX+$$
$$+\frac{\partial^2 U}{\partial x\partial y}(X,\ Y,\ \tau)\,q_yY-\dots \qquad (3.44)$$

Analogous resolution we will obtain also for derivative $\frac{\partial U}{\partial y}$. Let us substitute sums (3.43), (3.44) in equations (3.7) it is averaged equation for the period of focusing field. Taking into account equality (2.229) in exchange for equation (2.228) we will obtain

$$\frac{d^2X}{d\tau^2}-\left[\mu_0^2+\frac{eS^2}{\mathscr{E}_0\beta^2\gamma^3}\int_0^1 q_x(\tau)\frac{\partial^2 U}{\partial x^2}(X,\ Y,\ \tau)\,d\tau\right]X+$$
$$+\frac{eS^2}{\mathscr{E}_0\beta^2\gamma^3}Y\int_0^1 q_y(\tau)\frac{\partial^2 U}{\partial x\partial y}(X,\ Y,\ \tau)\,d\tau+$$
$$+\frac{eS^2}{\mathscr{E}_0\beta^2\gamma^3}\cdot\frac{\partial}{\partial x}\int_0^1 U(X,\ Y,\ \tau)\,d\tau=0.$$

We utilize for a comparative evaluation of members of the formulas, obtained under the assumption about the microcanonical distribution. Coulomb potential satisfies equation (3.18), whence

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=-\frac{I}{\pi\varepsilon_0\beta c r_x r_y}.$$

In the smooth approximation/approach, according to expression (3.29), for the matched beam we have

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\approx-\frac{I}{\pi\varepsilon_0\beta c K_c^2}(1-q_x-q_y).$$

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Since in smooth approximation/approach the beam is close to the axisymmetric,

$$\frac{\partial^2 U}{\partial x^2} \approx -\frac{I}{2\pi\varepsilon_0 \beta c R_c^2}(1 - q_x - q_y).$$

Hence

$$\frac{eS^2}{\varepsilon_0 \beta^2 \gamma^3}\int_0^1 q_x \frac{\partial^2 U}{\partial x^2}\, d\tau \approx \frac{r_a^2}{R_c^2}(\overline{q_x^2} - \overline{q_x q_y}),$$

where r_a — the characteristic Coulomb length, determined by formula (3.23). But in accordance with relationships/ratios (2.233), (3.32)

$$\frac{r_a}{R_c} \leqslant \mu_0; \ \overline{q_x^2} \ll 1; \ \overline{q_x q_y} \ll 1.$$

Thus, terms, which contain under the integral sign second derivatives that of potential, it is possible to disregard. The averaged potential $\overline{U}$ in the matched beam does not depend on the time

$$\overline{U}(X, Y) = \int_0^1 U(X, Y, \tau)\, d\tau.$$

As a result after averaging we have the following autonomous system of equations of the motion

$$\begin{aligned} &\frac{d^2X}{d\tau^2} + \mu_0^2 X + \frac{eS^2}{\varepsilon_0 \beta^2 \gamma^3} \cdot \frac{\partial \overline{U}}{\partial X}(X, Y) = 0; \\ &\frac{d^2Y}{d\tau^2} + \mu_0^2 Y + \frac{eS^2}{\varepsilon_0 \beta^2 \gamma^3} \cdot \frac{\partial \overline{U}}{\partial Y}(X, Y) = 0. \end{aligned} \qquad (3.45)$$

Potential $U(X, Y, \tau)$, undertaken at point X, Y, is determined by the equation of Poisson (3.18)

$$\frac{\partial^2 U}{\partial X^2}(X, Y, \tau) + \frac{\partial^2 U}{\partial Y^2}(X, Y, \tau) = -\frac{1}{\varepsilon_0}\varrho(X, Y, \tau). \qquad (3.18a)$$

After averaging equation (3.18a) for the period of focusing field we will obtain

$$\frac{\partial^2 \overline{U}}{\partial X^2} + \frac{\partial^2 \overline{U}}{\partial Y^2} = -\frac{1}{\varepsilon_0}\overline{\varrho}(X, Y). \qquad (3.46)$$

where

$$\bar{\varrho}(X, Y) = \int_0^1 \varrho(X, Y, \tau)\,d\tau.$$

Finally, averaging on the period of focusing field expression (3.1), we have

$$\bar{\varrho}(X, Y) = e \iint \bar{n}(X, Y, \dot{X}, \dot{Y})\,d\dot{X}\,d\dot{Y}. \qquad (3.47)$$

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Subsequently let us examine the autonomous dynamic system motion by which is described by equations (3.45). X, Y - generalized coordinates of this system. Let us name/call the dynamic system of that averaged indicated, and the corresponding to it model of bundle - by averaged beam. Of the sense of expression (3.47) $\bar{n}$ (X, Y, $\dot{X}$, $\dot{Y}$) - the distribution function in the phase space of the averaged beam. Thus, value $\bar{n}$ must depend only on the first integrals of equations (3.45). After solving together equations (3.45)-(3.47) when $\bar{n}$ - function of constants of motion, we will obtain the self-consistent solutions for the averaged beam. These solutions correspond to the self-consistent particle trajectories averaged on the period of focusing field in the real beam. If they are known to particle trajectory in the averaged beam X(τ), Y(τ), then the particle trajectories of the real beam are determined directly according to formulas (3.43).

Equations (3.45) have an integral of the type of the integral of the energy:

$$H(X, Y, \dot{X}, \dot{Y}) = \frac{1}{2}(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}\mu_0^2(X^2 + Y^2) + \frac{eS^2}{\mathscr{E}_0\beta^2\gamma^3}\bar{U}(X, Y). \tag{3.48}$$

If we substitute into expression (3.48) of equality (3.43), then we will obtain approximation for constant of motion of particles in the real beam

$$H(x, y, \dot{x}, \dot{y}, \tau) = \frac{1}{2(1+q_x)^2}\left[\left(\dot{x} - \frac{\dot{q}_x}{1+q_x}x\right)^2 + \mu_0^2 x^2\right] +$$
$$+ \frac{1}{2(1+q_y)^2}\left[\left(\dot{y} - \frac{\dot{q}_y}{1+q_y}y\right)^2 + \mu_0^2 y^2\right] + \frac{eS^2}{\mathscr{E}_0\beta^2\gamma^3}\bar{U}\left(\frac{x}{1+q_x}, \frac{y}{1+q_y}\right).$$

The obtained function $H(x, y, \dot{x}, \dot{y}, \tau)$ is not the Hamiltonian of reference system.

The potential averaged on the period of focusing field of the proper field of beam has axial symmetry. Let us switch over to the cylindrical coordinates

$$X = R\cos\Psi; \quad Y = R\sin\Psi.$$

Then

$$\bar{U}(X, Y) \equiv \bar{U}(R); \quad \bar{\varrho}(X, Y) \equiv \bar{\varrho}(R).$$

Let us introduce designation for the complete transverse particle momentum

$$P=\sqrt{\dot{X}^2+\dot{Y}^2}.$$

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Constant of motion (3.48) and the equation of field (3.46) can be rewritten in the form

$$H(R,\ P)=\frac{1}{2}P^2+\frac{1}{2}\mu_0^2R^2+\frac{eS^2}{\varepsilon_0\beta^2\gamma^3}\bar{U}(R); \qquad (3.49a)$$

$$\frac{d^2\bar{U}}{dR^2}+\frac{1}{R}\cdot\frac{d\bar{U}}{dR}=-\frac{1}{\varepsilon_0}\bar{\varrho}(R). \qquad (3.496)$$

The potential of proper field within the beam satisfies the boundary conditions

$$\bar{U}(0)=0;\ \frac{d\bar{U}}{dR}(0)=0. \qquad (3.50)$$

The functions of distribution (3.13), (3.41), (3.42) let us relate to the phase density of the averaged beam.

Let us examine, first of all, the microcanonical distribution

$$\bar{n}(R,\ P)=n_0\delta(H-H_0).$$

In this case

$$\bar{\varrho}(R)=2\pi en_0\int_0^\infty\delta(H-H_0)P\,dP.$$

But, according to expression (3.49a), with R=const we have PdP=dH, whence

$$\bar{\varrho}(R)=2\pi en_0\int_0^\infty\delta(H-H_0)\,dH=2\pi en_0.$$

The averaged density of space charge evenly distributed over the section of the averaged beam. Current in the beam

$$I = v \int\int \varrho(x, y, \tau)\, dx\, dy$$

on τ does not depend. After averaging on the period of focusing field we have

$$I = 2\pi v \int_0^{R_c} \bar{\varrho}(R)\, R\, dR. \qquad (3.51)$$

Hence

$$\bar{\varrho}(R) = \frac{I}{\pi v R_c^2} = \text{const.}$$

Substituting latter/last expression in equation (3.49b) and taking into account boundary conditions (3.50), we obtain the following solution for the averaged potential within the beam

$$\bar{U}(R) = -\frac{I}{4\pi\varepsilon_0 v}\left(\frac{R}{R_c}\right)^2.$$

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The equations of motion of particles in averaged beam (3.45) in this case prove to be linear and taking into account designation (3.23) are reduced to the form

$$\frac{d^2X}{d\tau^2} + \left(\mu_0^2 - \frac{r_a^2}{R_c^2}\right) X = 0;$$

$$\frac{d^2Y}{d\tau^2} + \left(\mu_0^2 - \frac{r_a^2}{R_c^2}\right) Y = 0.$$

The averaged solution in plane XOZ exists

$$X(\tau) = R_x \cos(\mu\tau + \Theta_x),$$
$$\frac{dX}{d\tau}(\tau) = -\mu R_x \sin(\mu\tau + \Theta_x),$$

where

$$\mu^2 = \mu_0^2 - \frac{r_a^2}{R_c^2}. \qquad (3.52)$$

Integral curves on planes X, $\dot{X}$ and Y, $\dot{Y}$ are ellipses. The boundaries of the averaged beam they reach particle with a maximum amplitude of averaged oscillations of $R_x = R_y = R_c$. To these particles corresponds on the phase plane the X$\dot{X}$ trajectory

$$\frac{X^2}{R_c^2} + \frac{\dot{X}^2}{\mu^2 R_c^2} = 1, \qquad (3.53)$$

covering the representative points of all particles. The product of the semi-axes of ellipse (3.53) is equal to $\bar{F}_0 = \mu R_c^2$. The reduced volume of real beam in the general case is determined by the expression

$$F_0 = T_0 \mu_r r^2.$$

Substituting in this expression of equality (2.235), (3.29), we obtain, that in the smooth approximation/approach $F_0 = \bar{F}_0$. Further, since $\mu = \frac{F_0}{R_c^2}$, that of equality (3.52) we have

$$\mu_0^2 - \frac{r_a^2}{R_c^2} - \frac{F_0^2}{R_c^4} = 0.$$

Hence a radius of the averaged beam is equal to

$$R_c = \sqrt{\frac{F_0}{\mu_0}} \sqrt{h + \sqrt{1 + h^2}}$$

where

$$h = \frac{r_a^2}{2\mu_0 F_0}$$

it is the Coulomb parameter of beam.

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Solution for a radius of the averaged beam coincides with solution (3.35) for the mean radius of matched beam, obtained above as a result of the averaging of strict self-consistent solutions in the real beam with uniform density distribution of space charge according to the section. Therefore all conclusions, which relate to the matched beams with smooth envelope, are applicable.

Let now the phase density of the averaged beam be distributed according to the degenerate law: $\bar{n}=n_0$ with $H\leqslant H_0$ and $\bar{n}=0$ with $H>H_0$. In accordance with integral expression (3.47)

$$\bar{\varrho}(R) = 2\pi e n_0 \int_0^{P_{\text{макс}}(R)} P\,dP$$

or

$$\bar{\varrho}(R) = \pi e n_0 [P_{\text{макс}}(R)]^2.$$

Since for any assigned R transverse impulse reaches maximum value/significance with $H=H_0$, then from expression (3.49a) we have

$$P^2_{\text{макс}} = 2H_0 - \mu_0^2 R^2 - \frac{2eS^2}{\mathscr{E}_0 \beta^2 \gamma^3} \bar{U}(R).$$

The density of space charge is distributed in the beam section according to the law

$$\bar{\varrho}(R) = \pi e n_0 \left[2H_0 - \mu_0^2 R^2 - \frac{2eS^2}{\mathscr{E}_0 \beta^2 \gamma^3} \bar{U}(R) \right]. \qquad (3.54)$$

Let us substitute expression (3.54) in the equation of averaged potential (3.49b). For the decrease of the subsequent recordings let us introduce the designation

$$R_0^2 = \varepsilon_0 \frac{\beta^2 \gamma^3 \mathscr{E}_0}{2\pi e^2 n_0 S^2}. \qquad (3.55)$$

Then equation (3.49b) is reduced to the form

$$\frac{d^2\bar{U}}{dR^2} + \frac{1}{R} \cdot \frac{d\bar{U}}{dR} - \frac{1}{R_0^2} \bar{U} = \frac{\beta^2 \gamma^3 \mathscr{E}_0}{eS^2 R_0^2} \left(\frac{1}{2} \mu_0^2 R^2 - H_0 \right).$$

Let us switch over to the dimensionless radius

$$s = \frac{R}{R_0}. \qquad (3.56)$$

In this case

$$\frac{d^2\bar{U}}{ds^2} + \frac{1}{s} \cdot \frac{d\bar{U}}{ds} - \bar{U} = \frac{\beta^2 \gamma^3 \mathscr{E}_0}{eS^2} \left(\frac{1}{2} \mu_0^2 R_0^2 s^2 - H_0 \right).$$

The solution of homogeneous equation, final in zero, is the modified Bessel function of zero-order $I_0(s)$ [86].

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The particular solution of nonhomogeneous equation takes the form

$$\bar{U} = \frac{\beta^2 \gamma^3 \mathscr{E}_0}{eS^2} \left(H_0 - 2\mu_0^2 R_0^2 - \frac{1}{2} \mu_0^2 R_0^2 s^2 \right).$$

Taking into account boundary conditions (3.50) we will obtain the following solution for the potential

$$\bar{U}(s) = \frac{\beta^2\gamma^3\mathscr{E}_0}{eS^2}\left[(2\mu_0^2R_0^2 - H_0)\{I_0(s) - 1\} - \frac{1}{2}\mu_0^2R_0^2s^2\right]. \quad (3.57)$$

From formulas (3.54), (3.57) follows the expression for density distribution of charge according to a radius of the averaged beam

$$\bar{\varrho}(s) = 2\varepsilon_0\mu_0^2\frac{\beta^2\gamma^3\mathscr{E}_0}{eS^2}\left[1 - \left(1 - \frac{H_0}{2\mu_0^2R_0^2}\right)I_0(s)\right]. \quad (3.58)$$

The density of space charge becomes zero on the boundary of averaged beam $R = R_c : \bar{\varrho}(k) = 0$, where $k = \frac{R_c}{R_0}$. Thus, for parameter k, which is determining a radius of averaged beam R_c, we have

$$I_0(k) = \frac{1}{1 - \frac{H_0}{2\mu_0^2R_0^2}}, \quad (3.59)$$

and expression (3.58) is reduced to the form

$$\bar{\varrho}(s) = 2\varepsilon_0\mu_0^2\frac{\beta^2\gamma^3\mathscr{E}_0}{eS^2}\left[1 - \frac{I_0(s)}{I_0(k)}\right]. \quad (3.58a)$$

Further, according to expression (3.51), beam current is equal to

$$I = 2\pi vR_0^2\int_0^k \bar{\varrho}(s)\,s\,ds.$$

Hence

$$I = \mu_0^2R_c^2\frac{\beta^3\gamma^3I_0}{2S^2}\left[1 - \frac{2I_1(k)}{kI_0(k)}\right]. \quad (3.60)$$

We will use expression (3.23) for characteristic Coulomb length:

$$1 - \frac{2I_1(k)}{kI_0(k)} = \frac{r_a^2}{\mu_0^2R_c^2}. \quad (3.61)$$

Is found one equation (3.61) for definition of two unknown values R_c, k from the assigned parameters of beam and channel r_a, μ_0. The second equation must be the expression, which connects values k and R_c with the transverse phase volume of beam. Let us note that the entering distribution (3.41) value n_0 was replaced with the dimensionless parameter k, to operate with which more conveniently.

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For the case of singular distribution we will obtain the equations of motion of particles in the averaged beam. Differentiating function (3.57), we have

$$\begin{aligned}\frac{\partial\bar{U}}{\partial X}&=\frac{\beta^2\gamma^3\mathscr{E}_0}{eS^2}\left[\frac{2I_1(s)}{sI_0(k)}-1\right]\mu_0^2X;\\ \frac{\partial\bar{U}}{\partial Y}&=\frac{\beta^2\gamma^3\mathscr{E}_0}{eS^2}\left[\frac{2I_1(s)}{sI_0(k)}-1\right]\mu_0^2Y.\end{aligned}\qquad(3.62)$$

Let us substitute these expressions in equations (3.45):

$$\begin{aligned}&\frac{d^2X}{d\tau^2}+\mu_0^2\frac{2I_1(s)}{sI_0(k)}X=0;\\ &\frac{d^2Y}{d\tau^2}+\mu_0^2\frac{2I_1(s)}{sI_0(Y)}Y=0.\end{aligned}\qquad(3.63)$$

The equations of transverse vibrations of particles in the intense beam with singular distribution of phase density prove to be nonlinear, since according to expression (3.56),

$$s=\frac{k}{R_c}\sqrt{X^2+Y^2}.$$

Above is obtained the expression for the potential of self-congruent field during stationary singular distribution (3.57). First integral (3.49a) it is possible to now write in the more specific form. Substituting potential (3.57) into constant of motion (3.49a) and utilizing equality (3.59), we obtain

$$H(s,\,P)=\frac{1}{2}P^2+H_0\frac{I_0(s)-1}{I_0(k)-1}\,.$$

Since for all particles of beam $H \leqslant H_0$, then

$$P^2 \leqslant 2H_0\left[1 - \frac{I_0(s)-1}{I_0(k)-1}\right].$$

Since $P^2 \geqslant 0$, then the particles, which reach the periphery of beam, corresponds constant of motion $H=H_0$. Transverse particle momentum, which reach the periphery of beam, is connected with the current radial displacement by the relationship/ratio

$$P^2 = 2H_0\left[1 - \frac{I_0(s)-1}{I_0(k)-1}\right].$$

To the periphery of beam s=k and the complete transverse impulse becomes zero. Constant of motion for the particles, which reach the periphery of beam, takes the form

$$\frac{1}{2}P^2 + \frac{H_0}{I_0(k)-1}I_0(s) = \frac{H_0}{I_0(k)-1}I_0(k).$$

Let us replace H_0 with its expression of equality (3.59)

$$\frac{1}{2}P^2 + 2\mu_0^2R_c^2\frac{I_0(s)}{k^2I_0(k)} = \frac{2\mu_0^2R_c^2}{k^2}.$$

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Thus, the representative points of peripheral particles lie in the four-dimensional phase space X, Y, $\dot{X}$, $\dot{Y}$ on the locked hypersurface

$$\frac{k^2}{4\mu_0^2 R_c^2}(\dot{X}^2+\dot{Y}^2)+\frac{1}{I_0(k)}I_0\left(k\frac{\sqrt{X^2+Y^2}}{R_c}\right)=1. \qquad (3.64)$$

This hypersurface encompasses the representative points of all particles in the four-dimensional phase space of the averaged beam and is, therefore, the boundary of the phase volume, occupied with beam. If during the microcanonical distribution the representative points are arranged/located only on the hypersurface, then in the case of singular distribution the representative points fill the limited four-dimensional volume. The projection of four-dimensional phase volume on plane X, $\dot{X}$ is determined with the aid of equations (2.310) and takes the form

$$\frac{k^2}{4\mu_0^2 R_c^2}\dot{X}^2+\frac{1}{I_0(k)}I_0\left(k\frac{X}{R_c}\right)=1. \qquad (3.65)$$

Projection coincides with the section of four-dimensional volume (3.64) by plane Y-0; $\dot{Y}$=0. The curve, described by equation (3.65), is given in Fig. 3.9. The semi-axes of this figure, according to expression (3.65), are respectively equal to

$$X_{\text{макс}} = R_c;$$
$$\dot{X}_{\text{макс}} = \frac{2\mu_0}{k} R_c \sqrt{1 - \frac{1}{I_0(k)}}.$$

Taking into account equality (3.59) we will obtain $\dot{X}_{\text{макс}} = \sqrt{2H_0}$.

The reduced volume of the real beam F_0 regarding exists divided into π the projected area of phase volume on plane $x, \frac{dx}{d\tau}$. Specifically, during this determination reduced volume is connected with the value of two-dimensional phase volume with relationship/ratio (2.113). In the smooth approximation/approach πF_0 it coincides with the projected area of the phase volume of the averaged beam on plane X, $dX/d\tau$

$$\pi F_0 = 4 \int_0^{R_c} \dot{X}\, dX.$$

The curve, described by equation (3.65), is not ellipse.

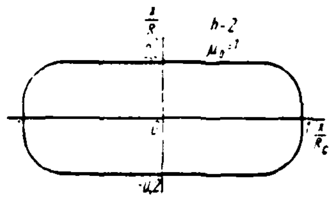

Fig. 3.8.

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Therefore reduced volume is not equal to the product of the semi-axes of projection and, according to equation (3.65), it comprises

$$F_0 = \frac{8\mu_0}{\pi k} R_c^2 \int_0^1 \sqrt{1 - \frac{I_0(k\xi)}{I_0(k)}}\, d\xi. \qquad (3.66)$$

Equality (3.66) - this is second missing equation for determining the values R_c, k. Let us introduce for the decrease of recording the following function from k:

$$\mathfrak{R}(k) = \frac{8}{\pi k} \int_0^1 \sqrt{I_0(k) - I_0(k\xi)}\, d\xi.$$

A radius of the averaged beam can be now written in the form

$$R_c = \sqrt{\frac{F_0}{\mu_0}} \sqrt{\frac{\sqrt{I_0(k)}}{\mathfrak{R}(k)}}. \qquad (3.67)$$

Let us substitute expression (3.67) in equation (3.61). This it gives

$$\frac{kI_0(k) - 2I_1(k)}{k\mathfrak{R}(k)\sqrt{I_0(k)}} = 2h, \qquad (3.68)$$

where h - introduced above Coulomb parameter of beam (3.34), proportional to phase current density. Thus, in terms of the given value of the Coulomb parameter it is possible to determine the auxiliary parameter k, with the aid of which is determined a radius of the averaged matched beam. Formulas (3.67), (3.68) together give the dependence of a radius of the averaged beam on the Coulomb parameter, analogous to simpler dependence (3.35), obtained for the microcanonical distribution.

The modified Bessel functions of zero and first order near k=0 take the form

$$I_0(k) = 1 + \frac{k^2}{4} + \frac{k^4}{64} + \dots;$$
$$I_1(k) = \frac{k}{2} + \frac{k^3}{16} + \dots$$

Hence with k<<1 it follows

$$\mathfrak{R}(k) = 1 + \frac{5}{128} k^2 + \dots;$$
$$R_c = \sqrt{\frac{F_0}{\mu_0}} \left(1 + \frac{11}{256} k^2 + \dots\right);$$
$$h = \frac{1}{16} k^2 + \dots$$

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Thus, value

$$R_c^0 = \sqrt{\frac{F_0}{\mu_0}} \tag{3.69}$$

is a radius of the averaged matched beam at the negligible phase current density. Formula (3.69) coincides in by expression (2.237). At the low values of phase current density the equation of curve (3.65), which limits the transverse volume of beam to plane X, $\dot{X}$, is reduced to the form

$$\frac{\dot{X}^2}{\mu_0^2 R_c^2}+(1-4h)\frac{X^2}{R_c^2}+h\frac{X^4}{R_c^4}=1-3h.$$

With $h \to 0$ curve (3.65) degenerates into the ellipse.

Since equations of motion (3.63) are nonlinear, then the frequency of transverse vibrations during singular distribution depends on amplitude. For paraxial particles s<<1 and from equations (3.63) we obtain

$$\frac{d^2X}{d\tau^2}+\frac{\mu_0^2}{I_0(k)}X=0.$$

The frequency of small transverse vibrations with $h \neq 0$ is equal to

$$\mu=\frac{\mu_0}{\sqrt{I_0(k)}}. \qquad (3.70)$$

The frequency of transverse vibrations of particles, which reach beam boundaries, is determined directly from the equation of phase trajectory (3.65):

$$\frac{1}{\mu_r}=\frac{1}{\mu_0}\cdot\frac{k\sqrt{I_0(k)}}{\pi}\int_0^1\frac{d\xi}{\sqrt{I_0(k)-I_0(k\xi)}}. \qquad (3.71)$$

At the low values of the phase current density of expression

(3.67), (3.70), (3.71) they take the form

$$R_c = R_c^0\left(1+\frac{11}{16}h-\ldots\right);$$
$$\mu=\mu_0(1-2h+\ldots),$$
$$\mu_r=\mu_0\left(1-\frac{5}{4}h+\ldots\right).$$

paraxial particles complete oscillations with the frequency of of lower than the particles, which reach beam boundaries.

From formula (3.68) it follows that parameter k monotonically increases with increase in h. In accordance with increase of h increases a radius of beam in the assigned focusing field. However, at any value/significance of h (in other words, at any value/significance of phase current density) a radius of beam retains finite value and, therefore, beam does not lose stability, if only the aperture of channel is sufficiently great.

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If k>>1, then function $I_0(k\xi)$ remains small in comparison with value $I_0(k)$ in the range $0\geq\xi>1$ up to the values ξ, very close to unity. Integrand in the formula, which is determining function $\mathfrak{N}(k)$, with increase of k graphically approaches a rectangle with the basis/base, equal to unity. Therefore function $\mathfrak{N}(k)$ at infinity behaves as

$$\mathfrak{N}(k)=\frac{8}{\pi k}\sqrt{I_0(k)}.$$

Function $I_1(k)$ with the increase of argument asimptotically tends for $I_0(k)$. Hence with k>>1

$$R_c = R_c^0 \sqrt{\frac{\pi k}{8}}; \quad h = \frac{\pi k}{16}. \tag{3.72}$$

Distribution according to a radius of the averaged density of three-dimensional/space charge in the case of singular distribution is described by formula (3.58a). Combining expressions (3.58a), (3.60), it is possible to represent the dependence of the charge density on a radius in the form

$$\bar{\varrho}\left(\frac{R}{R_c}\right) = \varrho_0 \left[1 - \frac{2I_1(k)}{kI_0(k)}\right]^{-1} \left[1 - \frac{I_0\left(k\frac{R}{R_c}\right)}{I_0(k)}\right]. \tag{3.73}$$

Here

$$\varrho_0 = \frac{I}{\pi v_s R_c^2}$$

is density of charge average over the section.

The canonical distribution of phase density (3.42) leads to following density distribution of space charge according to the section of averaged beam (3.47):

$$\bar{\varrho}(R) = 2\pi e n_0 \int_0^\infty e^{-\frac{H(R,P)}{\Theta}} P\, dP.$$

Let us substitute into this formula the general/common/total expression for constant of motion (3.49a):

$$\tilde{\varrho}(R) = 2\pi e n_0 \Theta \exp - \frac{1}{\Theta}\left(\frac{1}{2}\mu_0^2 R^2 + \frac{eS^2}{\mathscr{E}_0 \beta^2 \gamma^3}\bar{U}\right). \quad (3.74)$$

Taking into account expression (3.74) the equation of averaged potential (3.49b) takes the form

$$\frac{1}{R}\cdot\frac{d}{dR}\left(R\frac{d\bar{U}}{dR}\right) = -\frac{2\pi e n_0 \Theta}{\mathscr{E}_0}\exp - \frac{1}{\Theta}\left(\frac{1}{2}\mu_0^2 R^2 - \frac{eS^2}{\mathscr{E}_0 \beta^2 \gamma^3}\bar{U}\right)$$

Let us introduce instead of the averaged potential $\bar{U}(R)$ the new function

$$V(R) = V_0 \exp - \frac{1}{\Theta}\left(\frac{1}{2}\mu_0^2 R^2 + \frac{eS^2}{\mathscr{E}_0 \beta^2 \gamma^3}\bar{U}\right). \quad (3.75)$$

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According to expression (3.75),

$$\frac{d\bar{U}}{dR} = -\frac{\mathscr{E}_0 \beta^2 \gamma^3}{eS^2}\left(\frac{\Theta}{V}\cdot\frac{dV}{dR} + \mu_0^2 R\right). \quad (3.76)$$

If we utilize the dimensionless independent variable

$$s = \frac{\mu_0}{\sqrt{\Theta}} R,$$

then we will obtain the following equation for function V(s):

$$\frac{1}{s}\cdot\frac{d}{ds}\left(\frac{s}{V}\cdot\frac{dV}{ds}\right) + 2 = \frac{2\pi e^2 n_0 S^2 \Theta}{\mu_0^2 \varepsilon_0 m_0 c^2 \beta^2 \gamma^3 V_0} V(s).$$

Let us select now constant V_0 as follows

$$V_0 = \frac{2\pi e^2 n_0 S^2 \Theta}{\mu_0^2 \varepsilon_0 m_0 c^2 \beta^2 \gamma^3}. \quad (3.77)$$

Then equation for function V(s) is reduced to the form

$$\frac{1}{s}\cdot\frac{d}{ds}\left(\frac{s}{V}\cdot\frac{dV}{ds}\right)=V(s)-2. \qquad (3.78)$$

Initial conditions for function V(s) can be determined from expressions (3.75), (3.76) according to conditions (3.50), superimposed to the averaged potential

$$V(0)=V_0;\ \frac{dV}{ds}(0)=0. \qquad (3.79)$$

Density distribution of charge along a radius of the averaged beam is determined by equalities (3.74), (3.75)

$$\bar{\varrho}(s)=2\pi en_0\theta\frac{V(s)}{V_0}. \qquad (3.80)$$

The full current of beam (3.51) is equal to

$$I=\frac{4\pi^2 en_0 v\Theta^2}{\mu_0^2 V_0}\int_0^\infty V(s)\,s\,ds. \qquad (3.81)$$

Substituting in equality (3.81) for V_0 its expression (3.77) and utilizing designations for characteristic Coulomb length (3.23), we obtain

$$\int_0^\infty V(s)\,s\,ds=\frac{r_a^2}{\Theta}. \qquad (3.82)$$

During the canonical distribution of phase density the representative points are propagated over entire four-dimensional phase space. Therefore beam, generally speaking, does not have terminal radius.

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However, phase density rapidly drops during the removal/distance from the origin of coordinates, so that basic part of the beam current is concentrated in the region near the axis. Let us accept as a radius of averaged beam R_c a radius of the section, which covers η 100o/o of total beam current:

$$\int_0^{\mu_0 R_c^2/\sqrt{\Theta}} V(s)\, s\, ds = \eta \int_0^{\infty} V(s)\, s\, ds. \qquad (3.83)$$

R_c^0- radius of the averaged beam (in the sense indicated above) at the negligible phase current density. Then the reduced volume of beam F, it is expedient to define as the value, entering relationship/ratio (2.237):

$$R_c^0 = \sqrt{\frac{F_0}{\mu_0}}. \qquad (3.84)$$

Actually/really, in this case

$$\pi F_0 = \pi \mu_0 (R_c^0)^2$$

the area, occupied on plane by those X, $\dot{X}$ particles which give the contribution to the useful part of the full current of beam $\eta 100\%$,

$$V_n = \frac{\gamma}{cT_\Phi} F_0$$

the two-dimensional phase volume of this part of the beam. Introducing into equality (3.84) the dimensionless radius

$$s_0 = \frac{\mu_0}{1 \cdot \Theta} R_c^0,$$

we obtain

$$\Theta = \frac{\mu_0 F_0}{s_0^2}. \quad (3.85)$$

Relationship/ratio (3.82) takes the form

$$\int_0^\infty V(s)\, s\, ds = \frac{r_a^2}{\mu_0 F_0} s_0^2.$$

But combination the sublimity

$$h = \frac{r_a^2}{2\mu_0 F_0}$$

coincides with introduced at the analysis of other distributions Coulomb parameter of beam in the given focusing fields. Thus,

$$\int_0^\infty V(s)\, s\, ds = 2s_0^2 h. \quad (3.86)$$

Equality (3.86) is convenient to those that express the integral of the unknown function V(s) through the universal dimensionless parameter h.

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Different distributions it is expedient to compare between themselves at one and the same value/significance h, which does not depend on the type of distribution and which connects beam current, the phase volume of beam and lens power.

Value s_0 it is easy to determine. In the absence of space charge $\overline{U}(s)=0$. In this case, as it follows from equality (3.75),

$$V(s)=V_0 e^{-\frac{1}{2}s^2}.$$

After substituting this function into expression (3.83), we will obtain

$$\int_0^{s_0} e^{-\frac{1}{2}s^2} s\,ds=\eta\int_0^{\infty} e^{-\frac{1}{2}s^2} s\,ds.$$

Hence

$$s_0^2=-\ln(1-\eta)^2. \qquad (3.87)$$

Let us note that the value of integral (3.86) of selection s_0 does not depend, since function by V(s) its initial conditions (3.79) is determined unambiguously. However, arbitrary remains the selection of the limited value/significance of the phase volume of beam and, therefore, of Coulomb parameter of beam at the assigned parameters of beam and channel r_a, μ_0. If we define the two-dimensional projection of the phase volume of beam as region, where fall 90o/o of all particles, then $\eta=0.9$ and s_0=2.146. To virtually more conveniently conduct calculations, after assuming s_0=2; in this case in accordance with expression (3.87) $\eta=0.865$. The numerical calculations whose results are given below, are given for case of s_0=2. Thus, as a radius of the averaged matched beam was accepted a radius of the section, which covers 86.5o/o of the full current of beam.

Since value V_0 is actually unknown, the initial condition $V(0)=V_0$ must be replaced by integral condition (3.86). For each given value of the Coulomb parameter h function V(s) is determined by equation (3.78) and two supplementary conditions:

$$\int_0^\infty V(s)\, s\, ds = 2s_0^2 h; \quad \frac{dV}{ds}(0) = 0.$$

Now there can be found the fundamental characteristics of matched beam during the canonical distribution of phase density. Equalities (3.82), (3.86), (3.87) give equation for determining the mean radius of the beam

$$\int_0^{s_c(h)} V(s)\, s\, ds = 2s_0^2 \left(1 - e^{-\frac{1}{2}s_0^2}\right) h.$$

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Here

$$s_c(h) = \frac{\mu_0}{\sqrt{\theta}} R_c, \tag{3.88}$$

whence

$$R_c(h) = R_c^0 \frac{s_c(h)}{s_0}. \tag{3.89}$$

Finally, from expressions (3.80), (3.81), (3.86), (3.88) it is possible to find density distribution of charge according to the beam section for each given value of the Coulomb parameter:

$$\bar{\varrho}\left(\frac{R}{R_c}\right) = \varrho_0 \frac{s_c^2(h)}{4s_0^2 h} V\left(s_c \frac{R}{R_c}\right).$$

ρ_0 there is average density of charge in the averaged beam

$$\varrho_0 = \frac{I}{\pi v R_c^2}.$$

We investigate the behavior of function V(s). Since with increase in s the density of space charge $\overline{\rho}(s)$ must decrease, then with s>>1 we have V(s)<<1. Consequently, with s>>1 equation (3.78) is reduced to the following

$$\frac{d}{ds}\left(\frac{s}{V}\cdot\frac{dV}{ds}\right) = -2s.$$

The asymptotic behavior of function V(s) in infinity is described by the expression

$$V = V_1 s^{\alpha} e^{-\frac{1}{2}s^2}. \qquad (3.90)$$

On the other hand, substituting V(s) from equation (3.78) in equality (3.86), we obtain

$$2s_0^2 h = \int_0^{\infty}\left[2 + \frac{1}{s}\cdot\frac{d}{ds}\left(\frac{s}{V}\cdot\frac{dV}{ds}\right)\right] s\,ds$$

or

$$\left(s^2 + \frac{s}{V}\frac{dV}{ds}\right)_{s\to\infty} - \left(s^2 + \frac{s}{V}\frac{dV}{ds}\right)_{s\to 0} = 2s_0^2 h.$$

According to expression (3.79), second term on the left side is equal to zero. For first term we can use asymptotic expression (3.90). This it gives $\alpha = 2s_0^2 h$. Thus, with s>>1

$$V(s) = V_1 s^{2s_0^2 h} e^{-\frac{1}{2}s^2}.$$

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In the near-axial region s<<1 and $V(s) \approx V_0$. Equation (3.78) gives

$$\frac{d}{ds}\left(\frac{s}{V}\cdot\frac{dV}{ds}\right) = -(2-V_0)\,s,$$

whence

$$V = V_0 e^{-\left(1-\frac{1}{2}V_0\right)\frac{s^2}{2}}. \qquad (3.91)$$

Equation (3.78) has the particular solution $V(s) \equiv 2$. with V=2 integral (3.86) diverges and gives $h=\infty$. Thus, with a change of the Coulomb parameter within the limits $0<h<\infty$ value V_0 respectively is changed in the limits $0<V_0<2$. The behavior of function V(s) at the different values of the Coulomb parameter is shown in Fig. 3.9.

Let us determine the frequency of transverse vibrations of particles. Constant of motion (3.49a) taking into account equality (3.75) is reduced to the form

$$H(R,\,P) = \frac{1}{2}P^2 - \Theta \ln \frac{V(R)}{V_0}. \qquad (3.92)$$

According to expression (3.19), for the paraxial particles we have

$$H(R,\,P) = \frac{1}{2}P^2 + \frac{1}{2}\left(1-\frac{V_0}{2}\right)\mu_0^2 R^2,$$

or

$$\frac{1}{\mu_0^2\left(1-\frac{V_0}{2}\right)}P^2 + R^2 = \text{const.}$$

The projection of the phase trajectory of paraxial particle on each of the phase planes X, $\dot{X}$ or Y, $\dot{Y}$ is ellipse. The medium frequency of small transverse vibrations (to scale of axis τ) is equal to

$$\mu = \mu_0 \sqrt{1 - \frac{1}{2} V_0}\,.$$

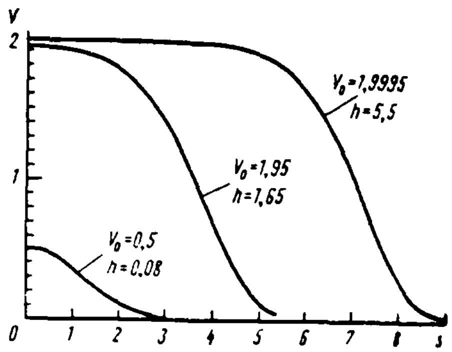

Fig. 3.9.

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At the low values of current density we can assume for any s

$$V(s) \approx V_0 e^{-\frac{1}{2}s^2}.$$

Substituting approximation for function V(s) into integral (3.86), we obtain

$$V_0 \int_0^\infty e^{-\frac{1}{2}s^2} s\,ds = 2s_0^2 h.$$

Consequently, with h<<1 we have $V_0 \approx 2s_0^2 h$ and

$$\mu \approx \mu_0 \left(1 - \frac{1}{2} s_0^2 h\right).$$

With s_0=2 this value coincides with the frequency of small transverse vibrations in the beam with the low phase current density during singular distribution.

Particles with the given value of constant of motion achieve maximum divergence from the axis with P=0. For the peripheral particles, which have the maximum amplitude of oscillations R_c from constant of motion (3.92) it follows

$$\frac{1}{2}P^2=\Theta\ln\frac{V(R)}{V(R_c)}.$$

Particle with coordinates $Y=\dot{Y}=0$ moves in plane X, $\dot{X}$ over the phase trajectory

$$\frac{1}{2}\left(\frac{dX}{d\tau}\right)^2-\Theta\ln\frac{V(X)}{V(R_c)}=0.$$

Hence can be obtained the medium frequency of transverse vibrations of peripheral particles during canonical distribution μ_r:

$$\frac{1}{\mu_r}=\frac{s_c\sqrt{2}}{\pi\mu_0}\int_0^1\frac{d\xi}{\sqrt{\ln\frac{V(s_c\xi)}{V(s_c)}}}.$$

Let us compare between themselves different cases of the stationary distribution of phase density.

In Fig. 3.10 is presented distribution of the averaged charge density over the beam section for the cases of the degenerate and canonical distributions. From the graphs/curves it is evident that at the low values of the Coulomb parameter density distribution of charge by a radius differs significantly from uniform. With an increase in the Coulomb parameter density distribution of charge in both cases approaches uniform. Hence it follows that at the high values of phase current density the parameters of beam must not depend either on the value of the phase volume of beam, or on the distribution of phase density.

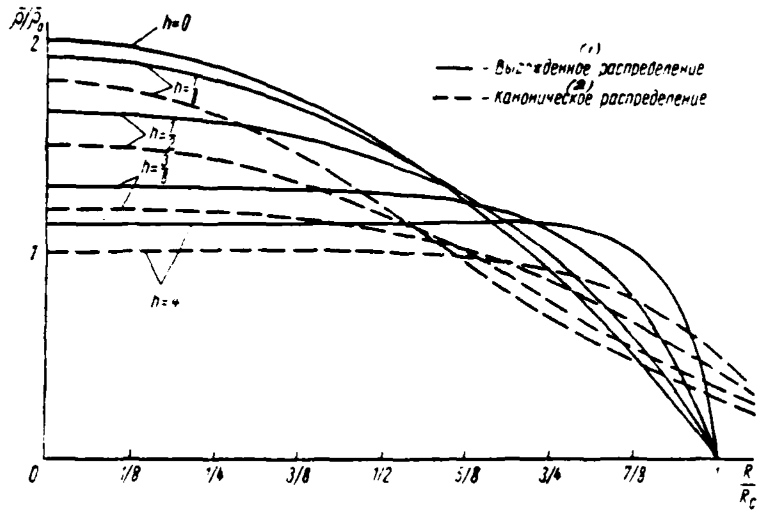

Fig. 3.10. Key: (1). Singular distribution. (2). Canonical distribution.

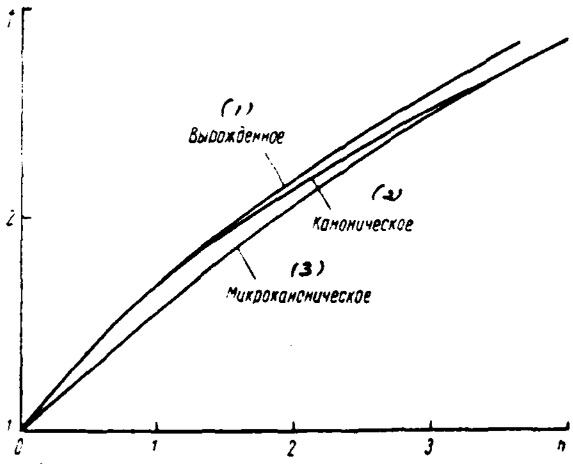

Fig. 3.11. Key: (1). Degenerated. (2). Canonical. (3). Microcanonical.

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Fig. 3.11 gives plotted functions

$$\frac{R_c}{R_c^0} = f(h) \qquad (3.93)$$

for three distributions of phase density. At the low values of the Coulomb parameter these functions have a somewhat different course; however in view of smallness h functions (3.93) do not manage to noticeably diverge. Since with $h \to \infty$ charge distribution according to the beam section approaches uniform, then functions (3.93) for different phase distributions must with $h \to \infty$ converge. Actually/really, with h>>1 for the microcanonical distribution from formula (3.35) we obtain

$$f(h) \approx \sqrt{2h + \frac{1}{2h}}. \qquad (3.94)$$

In the case of singular distribution from formulas(3.67), (3.68) it follows

$$f(h) = \sqrt{2h \frac{kI_0(k)}{kI_0(k) - 2I_1(k)}}.$$

If k>>1, then

$$f(h) \approx \sqrt{2h \cdot \frac{k}{k-2}}. \qquad (3.95)$$

Finally, in the case of canonical distribution with h>>1 we have V(s)=2 to $s > s_c$. From formulas (3.83), (3.86) we obtain

$$\int_0^{s_c} 2s\, ds = 2s_0^2 \eta h.$$

or $s_c^2=2s_0^2\eta h$. According to expression (3.89),

$$f(h)\approx\sqrt{\eta}\sqrt{2h}. \qquad (3.96)$$

At the sufficiently high values of the Coulomb parameter expressions (3.94), (3.95) coincide and give

$$f(h)\approx\sqrt{2h}. \qquad (3.97)$$

In the case of canonical distribution is obtained the somewhat smaller value in view of the fact that the mean radius of matched beam was determined for $\eta<1$. However, at the high values of h the distribution of charge along the radius has steep/abrupt shear/section. Assuming/setting with h>>1 value η of close one to unity, we obtain also for the canonical distribution formula (3.97) instead of (3.96). Thus, in entire range changes in the Coulomb parameter of function f(h) for all three distributions remain very close.

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During the degenerate and canonical distributions the equations of motion are nonlinear and the frequency of transverse vibrations depends on amplitude. One should in this case consider that are obtained the solutions for the stationary functions of distribution, so that the nonlinearity indicated does not lead to an increase in the equivalent phase volume of beam. Formally the retention/preservation/maintaining of phase volume is guaranteed by Liouville's theorem. In this case have in mind such changes in the forms of phase volume which could virtually lead to the result, equivalent to a change in its value.

The dependence of the medium frequencies of transverse vibrations on the Coulomb parameter for different distributions is given in Fig. 3.12. Along the axis of ordinates is deposited/postponed the ratio of medium frequency μ at this value/significance of the Coulomb parameter to value μ_0, of appropriate h=0. The thickened line relates to the linear oscillations, which correspond to uniform density distribution of charge according to the beam section. Dotted curves correspond to the frequencies of transverse vibrations of paraxial particles in the nonuniform distributions. With an increase in the Coulomb parameter of beam the frequency of particles, which move near the axis of bundle, rapidly decreases. From the point of view of the excitation of incoherent forced oscillations is important the frequency of peripheral particles. The medium frequencies of peripheral particles for the degenerate and canonical distributions are plotted/applied by continuous thin lines. With a limitless increase in the Coulomb parameter all curves in Fig. 3.12 must asymptotically pour.

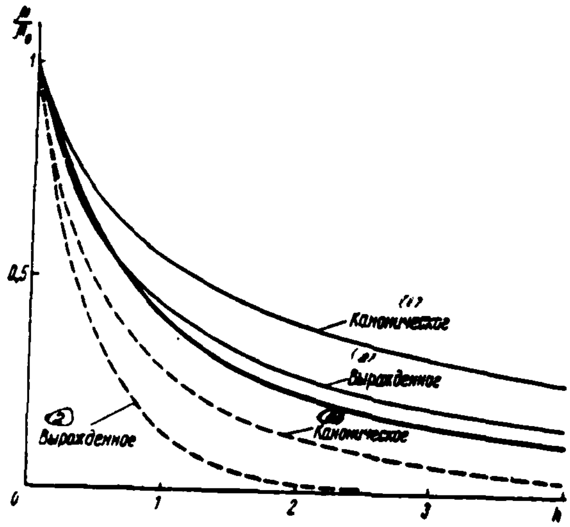

Fig. 3.12. Key: (1). Canonical. (2). Degenerate.

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From Fig. 3.12 it is evident that the frequency of particles, which reach the periphery of beam, during singular distribution virtually remains sufficiently close to the frequency of linear in entire range of a change in Coulomb parameter. During the canonical distribution the effect of transverse pushing apart on the frequency of particles, which reach periphery, proves to be less than highly expressed.

All stationary distributions examined with the accuracy of order 5-10o/o lead to one and the same numerical ratios between the sizes/dimensions of matched beam and the Coulomb parameter of beam in the assigned applied fields. Thus, the self-consistent solutions prove to be little sensitive to the form of the distribution of phase density in the four-dimensional phase space. In this sense four-dimensional phase volume proves to be the volume, which corresponds sufficiently to large number of degrees of freedom. Actually/really, let us examine four-dimensional sphere. If we divide in half a radius of this sphere, then within the sphere of a half radius will be located only 1/16 part of all particles during the even distribution. Rest 15/16 all particles will lie/rest at the external spherical layer. If we disregard/neglect the fields of the particles, which fall into the internal sphere, then it is evident that even distribution in the four-dimensional space must lead to the macroscopic parameters, close to the appropriate parameters at the microcanonical distribution.

The canonical distribution of phase density nearer than others corresponds to real beams. However, assumption about the microcanonical distribution leads to the simplest and easily forseeable relation between the parameters of beam and parameters of the focusing channel. From all that has been previously stated, it follows that the questions, connected with the effect of Coulomb pushing apart on transverse vibrations of particles, it suffices to examine, using formulas derived from the assumption about the stationary microcanonical distribution of phase density.

§ 3.5. Coherent oscillations of particles in the beam.

If the axis of bundle coincides with the axis of the focusing channel, then all phases of transverse vibrations are equally probable. The fluctuations of particles in this beam are incoherent. Beam is symmetrical relative to longitudinal axis in each of the planes XOZ and YOZ, and the center of the ellipse, which limits transverse phase volume, coincides since the origin of the coordinates on the phase plane (see Fig. 2.13). Errors in the focusing field (for example, displacement and the inclinations/slopes of the magnetic axes of quadrupole lenses), and also the error in the injection lead to the beam displacement as whole. In this case appear the oscillations of the center of gravity of beam (Fig. 3.13a). Appears coherent component transverse vibrations of particles.

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On each phase plane x, $dx/d\tau$ and y, $dy/d\tau$ the region, occupied with the representative points of the particles of the beam, is displaced from the origin of coordinates and rotates around the initial point (Fig. 3.13b). Let x_0, y_0 - coordinates of the center of gravity of beam, determined by the equalities

$$x_0 = \frac{1}{N}\sum_{i=1}^{N} x_i;$$

$$y_0 = \frac{1}{N}\sum_{i=1}^{N} y_i.$$

where the addition is conducted through all N to particles in this section. Let us note that in the equations of transverse vibrations (2.53) the coordinates of particle x, y are counted off from the magnetic axis of lens. Coordinates x, y, entering the Coulomb terms of equations (3.24), are counted off from the axis of bundle. In equations (3.24) the coordinates of particle x, y could be carried out as the signs of brackets only on the assumption that the axis of bundle coincides with the axis of channel. Let us introduce coordinates x, y, calculated off the axis of channel, and coordinate $\hat{x}$, $\hat{y}$. calculated off the axis of bundle,

$$\hat{x} = x - x_0; \quad \hat{y} = y - y_0.$$

Then expression for potential of the proper field of beam (3.21) will take the form

$$U = -\frac{1}{4\varepsilon_0}\varrho(\tau)\left[\hat{x}^2 + \hat{y}^2 - \right.$$
$$\left. -\frac{r_x - r_y}{r_x + r_y}(\hat{x}^2 - \hat{y}^2)\right] + \text{const.}$$

and equations of motion (3.24) can be rewritten in the form

$$\frac{d^2x}{d\tau^2} + Q_x(\tau)\,x - \frac{2r_a^2}{r_x(r_x + r_y)}(x - x_0) = 0;$$
$$\frac{d^2y}{d\tau^2} + Q_y(\tau)\,y - \frac{2r_a^2}{r_y(r_x + r_y)}(y - y_0) = 0.$$

If we write N of such equations for each of the particles of this section, sum up equations and sum divide into N, then we will obtain the following equations of motion of the center of gravity of the beam

$$\frac{d^2x_0}{d\tau^2}+Q_x(\tau)\,x_0=0;$$
$$\frac{d^2y_0}{d\tau^2}+Q_y(\tau)\,y_0=0. \qquad (3.98)$$

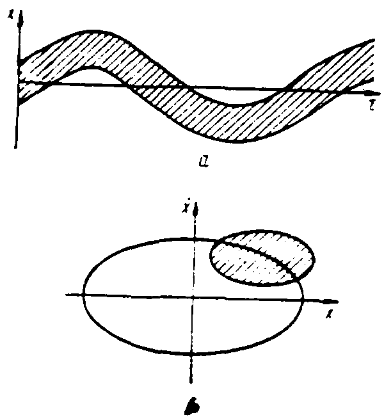

Fig. 3.13.

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Equations (3.98) coincide with the equations of transverse vibrations of particles in the beam with the negligible phase current density. Thus, in the free space the center of gravity of beam completes transverse vibrations with the frequency, for which the proper field of beam does not affect. Physically this is obvious, since Coulomb forces are internal and their resultant must be equal to zero.

As it was indicated above, the metallic surface of circular duct does not affect field within the axisymmetric beam, which is spread along the axis of duct. But if beam section is not circular, then in the proper field of beam appear the nonlinear terms, small in comparison with basic linear field component (during uniform density distribution of charge according to the section). Therefore the effect of metallic walls on the frequency of incoherent they disregarded/neglected. However, the dependence of the frequency of coherent oscillations on the phase current density can arise only due to the wall effect of channel, and therefore this effect cannot be rated/estimated disregarding by metallic walls.

Let us examine the beam of round cross section, displaced relative to the axis of circular duct with the ideally conducting walls (Fig. 3.14). The length of the period of coherent oscillations, as a rule, is substantially more than the transverse sizes/dimensions of beam. Therefore the potential of the proper field of beam can be found in the approximation/approach of two-dimensional problem. Without loss of generality we can consider that the beam is displaced only along the axis of abscissas. R_c - radius of beam; R_T - radius of the metal tube, within which is spread the beam; x_0 - center-of-gravity disturbance of beam relative to the axis of duct. It is assumed that the magnetic axis of focusing field is combined with the axis of duct. Let us introduce two polar systems. The center of the first polar system r, φ coincides with the axis of duct; the center of the second system of coordinates s, α coincides with the axis of bundle. Designations are clarified in Fig. 3.14. Between both coordinate systems are following relationships/ratios

$$s^2 = r^2 + x_0^2 - 2x_0 r \cos\varphi;$$
$$\operatorname{tg}\alpha = \frac{r\sin\varphi}{r\cos\varphi - x_0}. \qquad (3.99)$$

Let us assume first that the beam moves in free space

$(R_\text{т} = \infty)$.

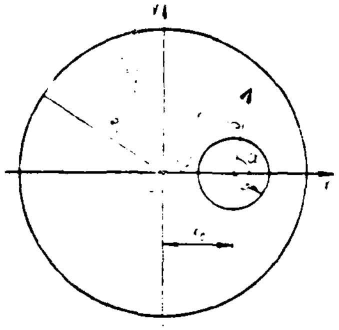

Fig. 3.14.

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Then the potentials of the proper field of beam satisfy the equations

$$\frac{d^2U_i}{ds^2}+\frac{1}{s}\cdot\frac{dU_i}{ds}=-\frac{1}{\varepsilon_0}\varrho;$$
$$\frac{d^2U_e}{ds^2}+\frac{1}{s}\cdot\frac{dU_e}{ds}=0, \quad (3.100)$$

where $U_i(s)$ - potential of field within the beam; $U_e(s)$ - potential of field out of the beam. Boundary conditions take the form

$$U_i(R_c)=U_e(R_c);\ \frac{dU_i}{ds}(R_c)=\frac{dU_e}{ds}(R_c). \quad (3.101)$$

From equations (3.100) and boundary conditions (3.101) it follows

$$U_i(s)=-\frac{\varrho}{4\varepsilon_0}s^2+U_0;$$
$$U_e(s)=-\frac{\varrho}{4\varepsilon_0}R_c^2\ln\left(\frac{s}{R_c}\right)^2-\frac{\varrho}{4\varepsilon_0}R_c^2+U_0.$$

Let now the beam of particles move within the circular ideally conducting ducts. V_i, V_e - respectively the potentials of proper field within the beam and in the space between beam and surface of duct. These potentials must satisfy boundary conditions on the surface of the beam

$$V_i(s,\alpha)_{s=R_c} = V_e(s,\alpha)_{s=R_c};$$
$$\frac{\partial V_i}{\partial s}(s,\alpha)_{s=R_c} = \frac{\partial V_e}{\partial s}(s,\alpha)_{s=R_c}; \qquad (3.102)$$
$$\frac{\partial V_i}{\partial \alpha}(s,\alpha)_{s=R_c} = \frac{\partial V_e}{\partial \alpha}(s,\alpha)_{s=R_c}$$

and on the surface of the duct

$$V_e(r,\varphi)_{r=R_T} = 0; \quad \frac{\partial V_e}{\partial \varphi}(r,\varphi)_{r=R_T} = 0. \qquad (3.103)$$

Let us represent potentials V_i, V_e in the form

$$V_i(r,\varphi) = U_i(s) + \Psi(r,\varphi);$$
$$V_e(r,\varphi) = U_e(s) + \Psi(r,\varphi),$$

where Ψ - certain harmonic function, which satisfies the equation of Laplace

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial\Psi}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2\Psi}{\partial\varphi^2} = 0. \qquad (3.104)$$

Then Poisson's equation for potential V_i, the equation of Laplace for potential V_e and boundary conditions (3.102) in accordance with equations (3.100), (3.101) are satisfied automatically. Boundary conditions (3.103) lead to the equalities

$$\Psi(r,\varphi)_{r=R_T} = -U_e(r,\varphi)_{r=R_T};$$
$$\frac{\partial\Psi}{\partial\varphi}(r,\varphi)_{r=R_T} = -\frac{\partial U_e}{\partial\varphi}(r,\varphi)_{r=R_T}. \qquad (3.105)$$

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Function $\Psi(r, \varphi)$ must be final on the axis duct, moreover in view of the obvious symmetry

$$\Psi(r, \varphi) = \Psi(r, -\varphi).$$

Dividing in equation (3.104) variable/alternating, we obtain the general solution in the form of the series/row

$$\Psi(r, \varphi) = \sum_{m=0}^{\infty} A_m r^m \cos m\varphi.$$

In order to satisfy boundary conditions (3.105), let us replace variable/alternating in function $U_e(s)$. According to expressions (3.99),

$$U_e(r, \varphi) = U_0 - \frac{\varrho R_c^2}{4\varepsilon_0}\left(1 - \ln\frac{r^2}{R_c^2}\right) - \frac{\varrho R_c^2}{4\varepsilon_0}\ln\left(1 - 2\frac{x_0}{r}\cos\varphi + \frac{x_0^2}{r^2}\right).$$

On the circle with a radius of R_T, with the center on the axis of the duct

$$U_e(R_T, \varphi) = U_0 - \frac{\varrho R_c^2}{4\varepsilon_0}\left(1 - \ln\frac{R_T^2}{R_c^2}\right) - \frac{\varrho R_c^2}{4\varepsilon_0}\ln\left(1 - \frac{2x_0}{R_T}\cos\varphi + \frac{x_0^2}{R_T^2}\right). \tag{3.106}$$

Latter/last function can be in turn, represented by Fourier series

$$U_e(R_T, \varphi) = \sum_{m=0}^{\infty} B_m(R_T)\cos m\varphi.$$

where with $m \neq 0$

$$B_m(R_T) = \frac{2}{\pi}\int_0^{\pi} U_e(R_T, \varphi)\cos m\varphi \, d\varphi.$$

After substituting here expression (3.106), we will obtain [28]

$$B_m(R_T) = \frac{\varrho R_c^2}{2\varepsilon_0 m}\left(\frac{x_0}{R_T}\right)^m.$$

Boundary conditions (3.105) are satisfied with

$$A_0 = B_0; \quad A_m = -\frac{\varrho R_c^2}{2\varepsilon_0 m R_\tau^m}\left(\frac{x_0}{R_\tau}\right)^m.$$

Hence

$$\Psi(r, \varphi) = B_0 - \frac{\varrho R_c^2}{2\varepsilon_0}\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r}{R_\tau}\right)^m\left(\frac{x_0}{R_\tau}\right)^m \cos m\varphi.$$

The potential of the proper field inside the beam, displaced relative to the axis of the ideally conducting duct, takes the form

$$V_i(r, \varphi) = -\frac{\varrho}{4\varepsilon_0}(r^2 - 2x_0 r\cos\varphi) - \frac{\varrho R_c^2}{2\varepsilon_0}\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r}{R_\tau}\right)^m\left(\frac{x_0}{R_\tau}\right)^m \cos m\varphi - + \text{const}.$$

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Intensity/strength of field at the center of gravity of the beam

$$E_x = -\frac{\partial V_i}{\partial r}(x_0, 0); \quad E_y = -\frac{1}{x_0}\frac{\partial V_i}{\partial \varphi}(x_0, 0),$$

or

$$E_x = \frac{\varrho}{2\varepsilon_0}\left(\frac{R_c}{R_\tau}\right)^2 x_0 \sum_{n=0}^{\infty}\left(\frac{x_0}{R_\tau}\right)^n; \quad E_y = 0. \qquad (3.107)$$

Analogous result was obtained in work [87] for the beam, which moves between two metallic planes.

Taking into account the conducting surface of duct the oscillations of the center of gravity of beam in plane XOZ are described at the nonrelativistic velocities by the equation

$$m_0 \frac{d^2x_0}{dt^2} = eE_x - F_x,$$

where F_x - focusing force, which acts on the particles in the center of gravity of beam. After transition to the independent variable τ we will obtain

$$\frac{d^2x_0}{d\tau^2} - \frac{S^2}{m_0v^2}F_x - \frac{eS^2}{m_0v^2}E_x = 0.$$

If beam is spread in the space, free from the metallic surfaces, then $R_\tau = \infty$ and $E_x = 0$. The oscillations of the center of gravity of beam in this case are determined by equations (3.98) or, in the smooth approximation/approach,

$$\frac{d^2x_0}{d\tau^2} + \mu_0^2 x_0 = 0.$$

Since is examined the averaged beam, one should assume

$$\frac{S^2F_x}{m_0v^2} = -\mu_0^2 x_0.$$

Let us substitute into formula (3.107) for the density of charge ρ its expression through the complete (peak) beam current

$$\rho = \frac{I}{\pi v R_c^2}$$

and we will be restricted to linear approximation to the proper field

$$E_x = \frac{I}{2\pi\varepsilon_0 v R_\tau^2} x_0.$$

Thus,

$$\frac{d^2x_0}{d\tau^2}+\left(\mu_0^2-\frac{eIS^2}{2\pi\varepsilon_0 m_0 v^3 R_\text{т}^2}\right)x_0=0.$$

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The recording of latter/last equation will be simplified, if we switch over to characteristic Coulomb length (3.23)

$$\frac{d^2x_0}{d\tau^2}+\left(\mu_0^2-\frac{r_\text{a}^2}{R_\text{т}^2}\right)x_0=0.$$

The frequency of coherent oscillations of beam as follows depends on a peak beam current (being determining parameter r_a) and a radius of the aperture of the channel

$$\mu_\text{кor}^2=\mu_0^2-\frac{r_\text{a}^2}{R_\text{т}^2}. \qquad (3.108)$$

Earlier was obtained expression (3.52) for the frequency of the incoherent of particles in the intense beam. Equalities (3.52) and (3.108) formally coincide, but they are distinguished by the fact that in the denominator of Coulomb term (3.52) stands the square of the mean radius of matched beam, and in expression (3.108) at the same place - square of a radius of the complete aperture of channel. Both frequencies with an increase in the intensity of beam decrease; however, differently. The frequency switch of incoherent oscillations depends on phase current density; explicitly this dependence is expressed by formula (3.39). The frequency switch of coherent oscillations on the phase volume of beam does not depend and is determined with the assigned aperture of channel only by peak beam current and by energy of particles. With an increase in the energy of particles decreases parameter r_a. Since the complete aperture of the focusing channel must be selected taking into account possible forced oscillations of beam as whole, then it exceeds the maximum amplitude of the free oscillations: $R_T > R_c$. Therefore the Coulomb frequency switch of coherent oscillations proves to be smaller than for the incoherent ones. In the beam with the essential phase current density occurs splitting/fission of the frequencies of transverse vibrations: the frequency of the coherent oscillations; it remains close to the frequency of particles in the beam with the zero intensity, and frequency of incoherent relatively rapidly decreases with an increase in the phase current density.

In § 3.3 is given a numerical example to beam matching with the strong-focusing channel of the linear accelerator of protons with the peak beam current I=400 ma and phase volume $V_n = 0.2$ cm·mrad. In the case μ_0=0.93 examined; h=0.68. Characteristic Coulomb length (3.23) with given speed of injection β=0.04, wavelength of accelerating field λ=2 m and period of focusing field S=$2\beta\lambda$ will comprise at the entrance of channel $r_a = 0.32$ cm. The mean radius of matched beam (3.35) is equal to $R_c = 0.4$ cm. The medium frequency of incoherent oscillations, calculated according to formula (3.39) or (3.52), proves to be equal to μ=0.48 and decreases almost two times in comparison with the frequency in the beam with the zero intensity. If a radius of complete aperture $R_т = 1$ cm, then the frequency of coherent oscillations (3.108) descends only to $\mu_{ког} = 0,87$.

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§ 3.6. Deliquescence of the drifting beam.

Phase current density - this is one of the most important parameters of those characterizing the quality of the beam, injected into the accelerator. For determining the phase current density it is necessary to measure the beam current and projection of four-dimensional phase volume on the phase planes (transverse phase volumes of beam). Different experimental installations, intended for measuring the transverse phase volumes, repeatedly were described in the literature [61, 62, 88-99]. In all these installations the phase volumes were determined on beam blowup, which drifts in the space, free from the applied fields. The deliquescence of the drifting beam it is connected with the disordered scatter of thermal velocities and with the action of its own coulomb field of beam. Therefore it is important to consider the effect of proper field to the parameters of the drifting beam or to utilize the measurement procedure which would eliminate this effect. The motion of intense beam in the space of free drift is examined in works [100-102]. However, in the works indicated it was assumed that the particle motion is determined only by the pushing apart Coulomb forces; the scatter of thermal velocities was considered as the negligible. Meanwhile the estimation of error, introduced by the transverse pushing apart of particles into the measurements of phase volumes, requires the simultaneous account of the phase volume of beam and its proper field.

The phase volume of beam it is possible to measure directly according to an increase in the transverse sizes/dimensions of beam on the assigned path of drift. But more accurate results can be obtained, if to consecutively/serially determine the deliquescence of the narrow beams, cut out from the complete beam with the aid of the special diaphragms. In the latter/last cases it is possible to find the distribution of phase particle density according to the volume, which is important for the establishment of the boundaries of the phase volume of real beam. Fig. 3.15 gives the installation diagram for measuring the phase volume by the method of "two slots". The first slot isolates all particles with the assigned coordinate x_0. The second slot makes it possible to establish/install the current distribution of the chosen particles according to the path inclinations dx/dz. After second diaphragm is established/installed the current-collecting device - faraday cylinder. Since both slots pass particles with all values of y, dy/dz, available in particles with the fixed/recorded coordinate x_0, after the circuit/bypass of all values x we obtain the projection of phase volume on plane x, dx/dz (emittance of beam). Method "four slots" is schematically shown in Fig. 3.16. The pair of slots in the first plane isolates particles with the given value of coordinates x_0, y_0. The second pair of slots makes it possible to measure the current distribution of the chosen particles according to the path inclinations dx/dz, dy/dz.

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If we place horizontal slot in the first plane to position $y_0=0$, and horizontal slot in the second plane to position dy/dz=0, then this method makes it possible to determine the section of four-dimensional phase volume by plane y=0, dy/dz=0. As it was shown in § 3.2, when the boundary of four-dimensional phase volume can be approximated by hyper-ellipsoid, the projection of phase volume on plane x, x coincides with the section by its coordinate plane y=y=0 [see equations (3.14), (3.15)].

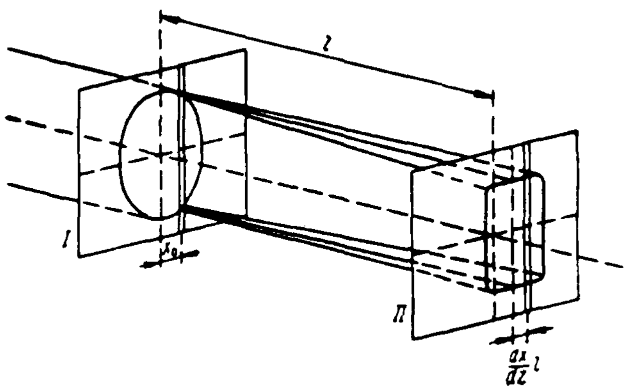

Fig. 3.15.

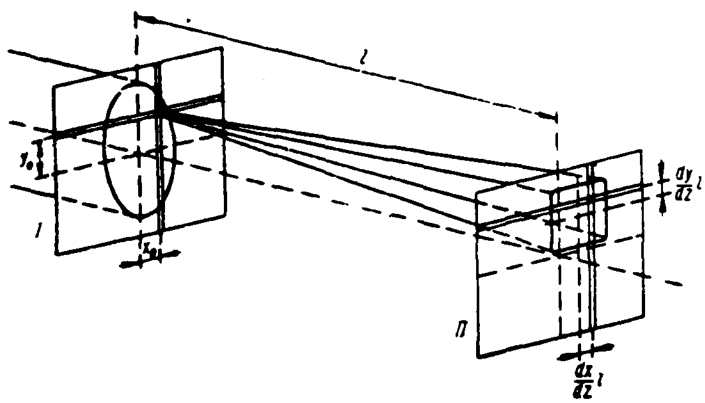

Fig. 3.16.

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Series circuit by both pairs of the slots of all points of the cross section of beam in the first and second planes it is possible to measure the distribution of phase density in the four-dimensional space.

Other methods of the consecutive irising of beam in principle are reduced to one of that described above. For example, for slot in the second plane can be replaced with light sensitive emulsion with the subsequent photometric measurement of the places exposed.

Let us examine particle motion on the section of the drift between two planes. Transverse phase volumes on planes x, x' and y, y' can prove to be unequal. In particular, this knowingly occurs in the diaphragmed beam with the method "two slots". The fact indicated must be taken into consideration in the equations for the envelopes. Particle motion in the drifting beam is described by equations (3.7), if we assume $Q_x(\tau) = Q_y(\tau) = 0$. Since the focusing fields are absent, parameter S it is possible to consider dimensionless and to place it equal to unity. Then as the independent variable will figure longitudinal length $\tau \equiv z$:

$$\frac{d^2x}{dz^2} + \frac{e}{\mathscr{E}_0\beta^2\gamma^3}\cdot\frac{\partial U}{\partial x} = 0;$$
$$\frac{d^2y}{dz^2} + \frac{e}{\mathscr{E}_0\beta^2\gamma^3}\cdot\frac{\partial U}{\partial y} = 0 \qquad (3.109)$$

Let the stationary distribution of phase density - microcanonical (3.13), so that current density is distributed evenly over the beam section, and equations (3.109) - linear, with the divided variable/alternating [30]. Since the independent variable in equations (3.109) is dimensional, let us change the conditions for the standardization of the dimensionless complex conjugate pairs of fundamental solutions (3.65)

$$\chi_x \frac{d\chi_x^*}{dz} - \chi_x^* \frac{d\chi_x}{dz} = -\frac{2i}{E_x};$$
$$\chi_y \frac{d\chi_y^*}{dz} - \chi_y^* \frac{d\chi_y}{dz} = -\frac{2i}{E_y}.$$

Values E_x, E_y must have a dimensionality of length. Let us represent fundamental solutions in the form (3.8). Rate of change of phase-fundamental solutions proves to be equal to

$$\frac{d\psi_x}{dz} = \frac{1}{E_x\sigma_x^2}; \quad \frac{d\psi_y}{dz} = \frac{1}{E_y\sigma_y^2}.$$

Having repeated the reasonings, given in § 3.2, we will obtain the following first integral of equations (3.109)

$$F\left(x, y, \frac{dx}{dz}, \frac{dy}{dz}\right) = \left(\sigma_x \frac{dx}{dz} - \frac{d\sigma_x}{dz} x\right)^2 + \left(\sigma_y \frac{dy}{dz} - \frac{\partial\sigma_y}{dz} y\right)^2 +$$
$$+ \left(\frac{x}{E_x\sigma_x}\right)^2 + \left(\frac{y}{E_y\sigma_y}\right)^2.$$

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Let us assume that the representative points of particles lie/rest on the hypersurface

$$F\left(x, y, \frac{dx}{dz}, \frac{dy}{dz}\right) = 1.$$

According to expressions (2.310), projection of this hypersurface on the phase planes they are limited by the ellipses

$$(\sigma_x x' - \sigma_x' x)^2 + \left(\frac{x}{E_x\sigma_x}\right)^2 = 1;$$
$$(\sigma_y y' - \sigma_y' y)^2 + \left(\frac{y}{E_y\sigma_y}\right)^2 = 1.$$

The products of the semi-axes of ellipses can be determined according to formula (2.114) and are equal respectively to E_x and E_y. In view of the fact that the ellipses are examined on planes x, dx/dz and y, dy/dz, values E_x, E_y are regarding the emittances of beam. Projection of hypersurface on plane x, y

$$\left(\frac{x}{E_x\sigma_x}\right)^2+\left(\frac{y}{E_y\sigma_y}\right)^2=1.$$

Hence it is apparent that envelope of particles are determined by the expressions

$$r_x(z)=E_x\sigma_x(z);\quad r_y(z)=E_y\sigma_y(z).$$

The linings/calculations, analogous to those given in § 3.2, 3.3, give the following equations for the envelopes

$$\begin{aligned}&\frac{d^2r_x}{dz^2}-\frac{E_x^2}{r_x^3}-\frac{2a^2}{r_x+r_y}=0;\\&\frac{d^2r_y}{dz^2}-\frac{E_y^2}{r_y^3}-\frac{2a^2}{r_x+r_y}=0.\end{aligned}\tag{3.110}$$

Parameter a in equations (3.110) is dimensionless and equal to

$$a=\sqrt{\frac{2I}{\beta^3\gamma^3I_0}},$$

where I - the full current of beam; I_0 - standard strength of current (3.22):

Let us examine first the pulverization/atomization of the non-stopped-down axisymmetric beam with emittances $E_x=E_y=E$ and initial conditions

$$\begin{aligned}&r_x(0)=r_y(0)=r_0;\\&\frac{dr_x}{dz}(0)=\frac{dr_y}{dz}(0)=r_0'.\end{aligned}$$

In view of the symmetry of equations (3.110) we have $r_x(z)\equiv r_y(z)=r(z)$, so that a change in the envelope of the axisymmetric beam is described by the equation

$$\frac{d^2r}{dz^2}-\frac{E^2}{r^3}-\frac{a^2}{r}=0. \qquad (3.111)$$

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Equation (3.111) is the generalization of the equation, investigated in works [64, 100-102], and are considered both space charge of beam and disordered scatter of the transversing speeds of particles in the beam. First integral of equation (3.111)

$$\left(\frac{dr}{dz}\right)^2=\left(\frac{E}{r_0}\right)^2\left(1-\frac{r_0^2}{r^2}\right)+a^2\ln\left(\frac{r}{r_0}\right)^2+(r_0')^2.$$

Hence we have

$$\frac{2E}{r_0^2}z=\int\limits_1^{(r,r_0)^2}\frac{ds}{\sqrt{\left[1+\left(\frac{r_0r_0'}{E}\right)^2\right]s+\left(\frac{ar_0}{E}\right)^2 s\ln s-1}}. \qquad (3.112)$$

Integral (3.112) determines the dependence of the current radius of the diffusing beam on the path of landing run z. In the literature on electron optics usually is examined case $E\ll r_0r_0'$, $E\ll ar_0$, corresponding to beam with the negligible scatter of transversing speeds. In this case integral (3.112) is reduced to well known expression [64]

$$\frac{z}{r_0}=\int\limits_1^{r/r_0}\frac{ds}{\sqrt{(r_0')^2+2a^2\ln s}}. \qquad (3.113)$$

Formula (3.112) makes it possible to now examine another limiting case - beam blowup with the negligible current. In the latter case of $a \ll E/r_0$ and from expression (3.112) it follows

$$\frac{r}{r_0}=\sqrt{\left(1+\frac{r_0'}{r_0}z\right)^2+\left(\frac{E}{r_0^2}\right)^2z^2}. \qquad (3.114)$$

With E=0 from equality (3.114) follows the trivial result

$$r=r_0-r_0'z.$$

The same result can be obtained from equality (3.113) with a=0.

If the space charge of beam is negligible, then formula (3.114) makes it possible to determine emittance and initial angular separation of beam. For the experimental determination of the values indicated ratio r/r_0 must be measured at two points: z_1 and z_2, after which it is possible to find both of the unknowns E and r'_0. In the general case for determining of emittance and initial angular separation of beam it is necessary to additionally know full current of beam and energy of particles, which makes it possible to previously calculate parameter a. After this of two equations (3.112) by numerical methods determine the unknowns r^2_0/E and $(r'_0)^2/E$. If with z=0 beam has a crossover, then it suffices to measure ratio r/r_0 only at one value/significance of z.

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The method of measurement of emittance indicated is less precise than the methods of the consecutive irising of beam.

Let us rate/estimate now space-charge effect on the disagreement of the separated part of the beam in the measurements of phase volume by the methods of consecutive irising. Let us assume that the measured beam possesses axial symmetry and its phase volume is limited in the four-dimensional space x, y, dx/dz, dy/dz by the hyper-ellipsoid

$$A(x^2+y^2)+B(x'^2+y'^2)+2C(xx'-yy')=1. \quad (3.115)$$

Beam section by plane XOY exists

$$x^2+y^2=R^2,$$

where

$$R=\frac{\sqrt{B}}{\sqrt{AB-C^2}}.$$

The emittance of beam for each of the phase planes

$$E=\frac{1}{\sqrt{AB-C^2}}$$

is connected with a radius of section R with relationship/ratio $R=E\sqrt{B}$. Let us isolate the part of the beam by narrow vertical slot with the width $2\Delta r$, also, with coordinate x_0 (see Fig. 3.15). The particles of the chosen part of the beam occupy on the phase plane x, x' the narrow band with a width of $2\Delta r$ (Fig. 3.17a). The maximum and minimum ordinates of band are approximately equal to the square roots equation

$$x'^2 + \frac{2Cx_0}{B}x' + \frac{Ax_0^2 - 1}{B} = 0,$$

whence

$$x' = -\frac{Cx_0}{B} \pm \frac{1}{\sqrt{B}}\sqrt{1 - x_0^2\left(A - \frac{C^2}{B}\right)}. \qquad (3.116)$$

The half-height of band is equal to

$$\Delta x' = \frac{1}{\sqrt{B}}\sqrt{1 - \left(\frac{x_0}{R}\right)^2}.$$

We approximate the chosen part of the emittance on plane x, x' by ellipse. Then the emittance of the chosen part of the beam is equal to E=ΔrΔx' or

$$E_x = E\frac{\Delta r}{R}\sqrt{1 - \left(\frac{x_0}{R}\right)^2}.$$

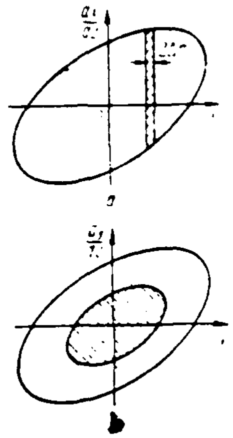

Fig. 3.17.

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Further, let us find the projection of the phase volume of the chosen part of the beam to plane y, y'. Particles with fixed/recorded coordinate x_0 fill the three-dimensional ellipsoid

$$F = Ay^2 + B(x'^2 + y'^2) + 2Cx_0x' + 2Cyy' = 1 - Ax_0^2.$$

Values x', which correspond to points in the curve, which covers the projection of this ellipsoid, are determined by equation $\frac{\partial F}{\partial x'} = 0$. Hence $x' = -(C/B)x_0$ and projection is described by the equation (see Fig. 3.17b)

$$A_1y^2 + B_1y'^2 + 2C_1yy' = 1,$$

where

$$A_1 = \frac{A}{1-\left(\frac{x_0}{R}\right)^2}; \quad B_1 = \frac{B}{1-\left(\frac{x_0}{R}\right)^2}; \quad C_1 = \frac{C}{1-\left(\frac{x_0}{R}\right)^2}. \quad (3.117)$$

Emittance of the chosen part of the beam on plane yy'.

$$E_y = E\left(1 - \frac{x_0^2}{R^2}\right).$$

Let us cut out now beam by supplementary narrow horizontal slot with the width 2Δr, also, with coordinate y_0 (see Fig. 3.16). According to expression (3.116) the half-height of band on plane y, y' will be equal to

$$\Delta y' = \frac{1}{\sqrt{B_1}} \sqrt{1 - y_0^2\left(A_1 - \frac{C_1^2}{B_1}\right)}.$$

Assuming/setting for the part of the beam, isolated with two mutually perpendicular slots, $E_{y1} = \Delta r \Delta y'$ and replacing A_1, B_1, C_1 by appropriate expressions (3.117), we will obtain the equality

$$E_{y1} = E\,\frac{\Delta r}{R} \sqrt{1 - \frac{x_0^2 + y_0^2}{R^2}}.$$

In view of axial symmetry $E_{x1} = E_{y1}$.

In the diaphragmed beam, isolated with two intersecting slots, emittances E_{x1}, E_{y1} are equal and for the evaluation of space-charge effect it is possible to use expression (3.112). The diaphragmed beam has the zero initial disagreement (see Fig. 3.17a), so that for it $r'_0=0$ and instead of integral (3.112) we obtain

$$\frac{2E_{x1}}{\Delta r^2}\,z = \int_1^{(r/\Delta r)^2} \frac{ds}{\sqrt{s - 1 + \left(\frac{a_1 \Delta r}{E_{x1}}\right)^2 s \ln s}} \qquad (3.118)$$

Parameter a_1 in interval (3.118) corresponds to the diaphragmed beam and in the constant current density in the section of complete beam is connected with parameter a of complete beam with the relationship/ratio

$$a_1^2 = a^2 \frac{\Delta r^2}{\pi R^2}.$$

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Hence

$$\frac{a_1 \Delta r}{E_{x_1}} = \frac{aR}{E} \cdot \frac{\Delta r}{R\sqrt{\pi}\sqrt{1 - \frac{x_0^2 - y_0^2}{R^2}}}.$$

The coefficient under the integral sign (3.118), connected with the Coulomb pushing apart, in the diaphragmed beam proves to be substantially less than in the complete beam. The effect of Coulomb pushing apart on the velocity of the deliquescence of the diaphragmed beam falls from the decrease of the width of slots, if only slots are not located on the edge of beam. Physically this is connected with the fact that the emittance of the chosen part of the beam decreases proportional to the width of slot (since with the irising far from the edge of phase volume $\Delta x' \sim$ const; see Fig. 3.17a), but beam current it falls proportional to the square of the width of slot. If slots are sufficiently narrow, then Coulomb pushing apart can be disregarded/neglected. This fact is the important advantage of the methods of the consecutive irising of beam.

Function s ln s calculation is more rapid than s-1. Therefore in proportion to beam blowup the effect of Coulomb pushing apart is amplified. Let it be the drift of the stopped-down beam for reasons of the accuracy of measurements it must be as more as possible. However, to disregard/neglect the effect of the proper field of beam is possible only in such a case, when the path of drift is limited.

$$s_{\text{макс}} = \left(\frac{r_{\text{макс}}}{\Delta r}\right)^2.$$

Coulomb pushing apart can be disregarded/neglected, if

$$\left(\frac{a_1 \Delta r}{E_{x_1}}\right)^2 s_{\text{макс}} \ln s_{\text{макс}} \ll s_{\text{макс}} - 1. \qquad (3.119)$$

Requirements for the accuracy of measurements lead to condition $s_{\text{макс}} \gg 1$. Then from inequality (3.119) it follows

$$\ln \frac{r_{\text{макс}}}{\Delta r} \ll \frac{\pi}{2} \left(\frac{E}{a \Delta r}\right)^2 \left(1 - \frac{x_0^2 + y_0^2}{R^2}\right). \qquad (3.120)$$

Let now the diaphragmed beam be isolated only with one slot, for example vertical. The initial sizes/dimensions of envelopes are equal to

$$r_x(0) = \Delta r; \quad r_y(0) = R \sqrt{1 - \left(\frac{x_0}{R}\right)^2}.$$

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If slot is not located on the edge of beam, then the chosen part is the strip/tape beam which has $r_x(0) \ll r_y(0)$. On the limited path of drift will be preserved inequality $r_x \ll r_y$ and a change in the envelopes will be described approximately by the equations

$$\frac{d^2r_x}{dz^2}-\frac{E_x^2}{r_x^3}-\frac{2a_2^2}{r_y}=0;$$
$$\frac{d^2r_y}{dz^2}-\frac{E_y^2}{r_y^3}-\frac{2a_2^2}{r_y}=0, \tag{3.121}$$

where

$$a_2^2=a^2\,\frac{r_x(0)\,r_y(0)}{\pi R^2}.$$

Let us replace in equations (3.121) the variable/alternating

$$u_x(z)=\frac{r_x(z)}{r_x(0)};\quad u_y(z)=\frac{r_y(z)}{r_y(0)}.$$

In the designations

$$e=\frac{E_x}{[r_x(0)]^2};\quad k=\frac{r_x(0)}{r_y(0)};\quad b^2=\frac{2a^2}{\pi R^2}$$

of equation (3.121) let us rewrite in the form

$$\frac{d^2u_x}{dz^2}-\frac{e^2}{u_x^3}-\frac{b^2}{u_y}=0;$$
$$\frac{d^2u_y}{dz^2}-\frac{k^2e^2}{u_y^3}-\frac{kb^2}{u_y}=0, \tag{3.122}$$

moreover $u_x(0)=u_y(0)=1;\ k\ll 1$. From equations (3.122) it is evident that a relative change of the envelope in the vertical plane is considerably less than in the horizontal, since second equation (3.122) corresponds to beam with the relatively small phase volume and the low current. In other words, strip/tape beam diffuses so that in essence occurs the relative expansion of its narrow side. Therefore for further estimations we can assume $u_y(z)\approx 1$. In the approximation/approach indicated first equation (3.122) gives

$$\left(\frac{du_x}{dz}\right)^2=e^2\left(1-\frac{1}{u_x^2}\right)+2b^2(u_x-1).$$

Toward the end of the section of drift $u_x = u_{\text{макс}}$. The effect of the proper field of the diaphragmed beam can be disregarded/neglected, if

$$2b^2 (u_{\text{макс}} - 1) \ll e^2 \left(1 - \frac{1}{u^2_{\text{макс}}}\right).$$

After assuming $r_{x_{\text{макс}}} \gg \Delta r$ or $u_{\text{макс}} \gg 1$, we will obtain

$$u_{\text{макс}} \ll \frac{e^2}{2b^2}.$$

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Hence

$$\frac{r_{\text{макс}}}{\Delta r} \ll \frac{3}{4} \left(\frac{E}{a \Delta r}\right)^2 \left(1 - \frac{x_0^2}{R^2}\right). \qquad (3.123)$$

For the non-stopped-down beam the condition under which it is possible to disregard the effect of proper field to beam blowup, follows directly from integral (3.112). Let at the initial point of drift the beam have a crossover with a radius of r_0=R. Coulomb pushing apart can be disregarded/neglected, until beam blowup exceeds the values, determined by order of value by the inequality

$$\ln \frac{r_{\text{макс}}}{R} \ll \frac{1}{2} \left(\frac{E}{aR}\right)^2. \qquad (3.124)$$

Condition (3.123) is harder/more rigid than condition (3.120). The method of four slots makes it possible to suppress the effect of the proper field of beam better than this is reached by the method of two slots. Let us examine a numerical example. Let the measured beam have transverse phase volume, approximately equal to $V_n = 0.2$ cm•mrad, with the full current I=200 mA and at the given particle speed β=0.04. Let us accept the width of each slot of the equal to 1 mm, a radius of beam in the crossover 1 cm. Then E=5 cm•mrad; $a=6.3\cdot10^{-3}$; E/aR=0.8; E/aΔr=16. For the non-stopped-down beam condition (3.124) gives

$$\ln\frac{r_{макс}}{R} \ll 0.32; \quad \frac{r_{макс}}{R} \ll 1.4.$$

Thus, to determine phase volume on the deliquescence of the non-stopped-down beam without taking into account Coulomb pushing apart in the assigned parameters is virtually inadmissible. For the beam, isolated with one slot, we will obtain

$$\frac{r_{макс}}{\Delta r} \ll 200\left(1-\frac{x_0^2}{R^2}\right).$$

If slot is displaced to position $x_0=0.9R$, then $\frac{r_{макс}}{\Delta r} \ll 40$, which makes it possible to still conduct measurements with the necessary accuracy, without taking into account three-dimensional charge of beam. For the beam, isolated with two slots, we have

$$\ln\frac{r_{макс}}{\Delta r} \ll 400\left(1-\frac{x_0^2+y_0^2}{R^2}\right).$$

or with $\sqrt{x_0^2+y_0^2}=0.9R$.

$$\ln\frac{r_{макс}}{\Delta r} \ll 80.$$

Latter/last inequality virtually does not limit the permissible expansion of the diaphragmed beam.

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§ 3.7. Limitations of beam current in the strong-focusing linear accelerator.

The mean radius of matched beam in the strong-focusing channel of linear accelerator grows/rises with an increase in the transverse phase volume of beam and with an increase in the peak beam current. Since the aperture of channel is limited, proves to be limited the maximum peak beam current in the assigned fields. The ratio of the mean radius of matched beam with the given value of the Coulomb parameter $h \neq 0$ to the mean radius of the beam of zero intensity with the same transverse phase volume is determined by formula (3.93). Multiplying numerator and denominator on the left side of equality (3.93) at the maximum value/significance of the modulating function $1+q(\tau)$, we will obtain that in the smooth approximation/approach function (3.93) determines the relation of the maximum values of the envelopes:

$$\frac{r_{\text{макс}}}{r^0_{\text{макс}}} = f(h).$$

Beam is passed without the losses through the focusing channel, if $r_{\text{макс}} \leqslant a$, where a - radius of the aperture, diverted under the free fluctuations of particles. Beam current reaches the maximum permissible value/significance when $r_{\text{макс}} = a$. Hence

$$r^0_{макс} = \frac{a}{f(h)}. \qquad (3.125)$$

By latter/last equality is assigned the maximum size of the matched beam of zero intensity, which has the same transverse phase volume, as the beam of the maximum permissible intensity. The transverse phase volume of the beam of zero intensity is determined by formula (2.125) and can be represented in the form

$$V_a = \frac{1}{c}\gamma\omega^0_{r\ мин}(r^0_{макс})^2,$$

where $\omega^0_{r\ мин}$ - minimum value/significance of the frequency of transverse vibrations of particles in the beam of zero intensity. After substituting expression (3.125) into the latter/last formula, we will obtain the maximum phase volume of the beam which can be passed through the focusing channel at the value/significance of the Coulomb parameter h≠0

$$V_{кh} = \frac{\gamma\omega^0_{r\ мин}a^2}{cf^2(h)}$$

or

$$V_{кh} = \frac{V_к}{f^2(h)},$$

where $V_к$ - channel capacity (2.126).

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In the principle the behavior of function f(h) depends on the form of the stationary distribution of phase density. However, as was shown in § 3.4, the behavior of this function depends on the form of distribution very weakly and for further estimations it is possible to utilize function (3.35), obtained for the microcanonical distribution

$$f(h) = \sqrt{h} - \sqrt{1-h^2}.$$

Thus,

$$V_{\kappa h} = V_{\kappa}\left(\sqrt{1-h^2} - h\right). \qquad (3.126)$$

Since the Coulomb parameter is proportional to phase current density, with an increase in the phase current density of beam the maximum permissible phase volume of the beam, passing through the channel without the losses, decreases.

Let us assume that the transverse phase volume of the beam, injected into the accelerator, is lower than the capacity of the focusing channel of accelerator. If phase current density in the beam is small, then will be fulfilled also inequality $V_{\text{п}} < V_{\kappa h}$. From an increase in the phase current density value $V_{\kappa h}$ falls. The maximally high value/significance of phase current density, which corresponds $h = h_{\text{макс}}$, will be achieved/reached when $V_{\kappa h} = V_{\text{п}}$:

$$V_{\text{п}} = V_{\kappa}\left(\sqrt{1-h^2_{\text{макс}}} - h_{\text{макс}}\right).$$

Hence follows

$$h_{\text{макс}} = \frac{1}{2}\left(\frac{V_{\kappa}}{V_{\text{п}}} - \frac{V_{\text{п}}}{V_{\kappa}}\right). \qquad (3.127)$$

Expression (3.127) determines a maximally possible peak beam current at the given values of the channel capacity and transverse phase volume of beam. Taking into account designation (3.36) we will obtain

[103]

$$I_{\text{макс}} = \frac{1}{2c}\beta\gamma^2\Omega_r^2 I_0 V_к \left(1 - \frac{V_п}{V_к} \text{[illegible]}\right).$$

With the assigned energy of particles and in the assigned parameters of channel a maximally possible current decreases with an increase in the phase volume of beam. An increase in the current of injection $I_{\text{инж}}$ makes sense as long as $I_{\text{инж}} < I_{\text{макс}}$. If an increase in the current of injection is accompanied by a simultaneous increase in the transverse phase volume of beam, then maximally possible current $I_{\text{макс}}$ descends. In this case the current of injection can prove to be not used completely. With an increase in the current of injection they increase in essence only of the loss of particles in the accelerator. Hence it is apparent that an increase in the current of injection is effective only in such a case, when simultaneously there does not occur a considerable increase in the transverse phase volume of beam. A considerable increase in the current of injection must be accompanied by the appropriate increase in the phase current density of beam.

Maximum peak current is reached when $V_п < V_к$ and it is equal to

$$I_{\text{пред}} = \frac{\beta\gamma^2\Omega_r^2}{2c} V_к I_0. \qquad (3.128)$$

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Taking into account expression (3.128) we have

$$I_{\text{макс}} = I_{\text{пред}}\left[1 - \left(\frac{V_{\text{п}}}{V_{\text{к}}}\right)^{2/3}\right]. \qquad (3.129)$$

A maximally possible peak current is virtually equal to maximum, if the transverse phase volume of beam approximately/exemplarily three times lower than channel capacity. Further reduction in the phase volume of beam, if it can be realized, already insignificantly affects the strength of maximally possible current of the accelerated beam. On the other hand, the increase in the Coulomb parameter, connected with an increase in the phase current density, leads to unjustified lowering of frequency of incoherent.

An increase in the amplitude of transverse vibrations due to the random errors in the focusing fields is approximately determined by expression (2.280):

$$\langle \Delta A^2 \rangle \approx \frac{N_\Phi}{2\nu_\Phi^2} \sum \langle \Delta \dot{x}^2 \rangle, \qquad (3.130)$$

where $\nu_\Phi = \omega_{r\,\text{мин}} T_\Phi$, and sum is taken on the basis of all independent sources of errors in each period of focusing field. The disturbances/perturbations of path inclination in each lens, in view of the fact that the disturbance/perturbation occurs in the small section of longitudinal axis, do not depend on the forces of Coulomb pushing apart. Actually/really, after replacing envelopes in the equations of motion (3.24) by appropriate expressions (3.29a) and disregarding small components/terms/addends q_x, q_y, we will obtain

$$\frac{d^2x}{d\tau^2} + \left[Q - \left(\frac{r_a}{R_c} \right)^2 \right] x = 0 \qquad (3.131)$$

Equation (3.131) it is possible to use for describing the particle motion in the section of one lens with Q=const. From equalities (3.32), (3.34) it follows

$$\left(\frac{r_a}{R_c} \right)^2 = \mu_0^2 \frac{2h}{\sqrt{1-h^2}-h}.$$

Relation $\left(\frac{r_a}{R_c} \right)^2$ monotonically is changed from zero (with h=0) to μ^2_0 with $h \to \infty$. Thus, always

$$\left(\frac{r_a}{R_c} \right)^2 < \mu_0^2.$$

But in the smooth approximation/approach, according to expression (2.229a), $\mu^2_0 \sim \overline{q}Q$ and, since $|q| \ll 1$, we have $\mu^2_0 \ll Q$. Hence

$$\left(\frac{r_a}{R_c} \right)^2 \ll Q.$$

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If oscillations are examined in the short section of change τ, then it is possible to disregard the Coulomb term of equation (3.131). The disturbances/perturbations of oscillations in the section of each lens can be calculated on the assumption that the phase current density is negligible. Space-charge effect on forced oscillations of particles is determined only by the frequency switch of oscillations. Let us divide sum (3.130) into two parts

$$(\Delta A)^2 = \frac{N_\psi}{2(\nu_\psi)^2} \sum (\Delta \dot{x})^2 - \frac{N_\psi}{2(\nu_\psi^*)^2} \sum (\Delta \dot{x})^2.$$

Into the first sum enter the disturbances/perturbations, connected with the external errors (displacement and the inclination/slope of the magnetic axes of lenses), and into the second - disturbances/perturbations, connected with the parametric errors (scatter of the gradients of focusing fields, the rotation of median axes). External errors act on all particles they equally and displace beam as whole. The effect of external errors depends on the frequency of coherent oscillations ν_{Φ}. Parametric errors lead to incoherent forced oscillations with minimum frequency ν^{*}_{Φ}, of that connected with $\nu^{0}_{\text{мин}}$ the relationship/ratio, analogous (3.38),

$$\nu^{*}_{\Phi} = \delta^{0}_{\text{мин}} \left(\sqrt{1-h^{2}} - h \right) .$$

The frequency of coherent oscillations weakly depends on the space charge of beam, but frequency of incoherent with an increase in the Coulomb parameter substantially descends. The portion of incoherent disturbances/perturbations grows/rises, as which proves to be necessary to stiffen either all structural/design and electrical allowances or only allowances for the parametric errors. Thus, a considerable reduction in the frequency of incoherent is undesirable and, therefore, is undesirable an excessive increase in the phase current density. From that presented it is evident that the relationship/ratio

$$V_{\text{п}} \approx \frac{V_{\text{к}}}{3}$$

is optimum. It should be noted that when $I = I_{\text{макс}}$ and $V_{\text{п}} = 1/3\, V_{\text{к}}$ the allowances for the parametric errors in the initial part of the accelerator descend approximately/exemplarily three times.

Maximally possible peak beam current in the focusing channel is higher, the greater the value of limiting current (3.128). Limiting current depends only on the parameters of accelerator and it is not connected with the quality of beam. For the analysis of formula (3.128) the latter it is convenient to convert, utilizing expressions (2.106), (2.126), which are determining the medium frequency of transverse vibrations and channel capacity

$$I_{\text{пред}} = \frac{\mu_0 \nu_\Phi^0}{2}\left(\frac{a}{S}\right)^2 \beta^3\gamma^3 I_0. \qquad (3.132)$$

We investigate the basic methods of increasing the limiting current of beam.

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With the assigned structure of channel values μ_0 and ν_Φ^0 are connected unambiguously, since μ_0 - the medium frequency of transverse vibrations to scale τ, and ν_Φ^0 - minimum value/significance of this frequency on the same scale. The parameter μ_0 in the limits of stability region has certain optimum value, which ensures value ν_Φ^0, close to the extreme, with the smallest expenditure of power for focusing. Optimum selection noncritically depends on the structure of channel. Thus, product $\mu_0\nu_\Phi^0$ is determined during the optimum selection virtually unambiguously.

Thus, limiting current it is possible to affect only by a change in the energy of injection W_0 and ratio of the aperture of accelerator to the period of focusing field. In the assigned ratio $S/\beta\lambda$ we have $I_{\text{пред}} \doteqdot \beta\gamma^3$. With the energy of particle injection into the ionic accelerator Lorenz's factor remains still in effect equal to unity, so that limiting current is proportional to square root from the energy of the injection

$$I_{\text{пред}} \doteqdot \sqrt{W_0}.$$

An increase in the energy of injection is hindered/hampered due to the complication of operation and rise in price of the injector, as which usually is utilized the electrostatic accelerator.

Further, limiting current increases proportional to the square of the ratio of the aperture of channel to the period of focusing field. In the accelerating systems with drift tubes with an increase in the aperture deteriorates the configuration of high-frequency field in the accelerating clearances and descends the factor of transit time. However, this fact is not deciding during the selection of aperture. Thus, in the majority of the acting proton accelerators it would be possible to still considerably increase aperture without a noticeable reduction in the factor of transit time. More essential prove to be the limitations, connected with an increase of the magnetic core induction of quadrupole lenses with the assigned gradients of focusing field. In the magnetic quadrupole lenses is limited maximum induction $B_{\text{макс}}$ in the poles, generatrices to Linz. Analogous limitations to the maximum stress/voltage between the electrodes lenses. Let us examine in more detail the case of magnetic lenses. The maximum gradient of the focusing field

$$G_{\text{макс}} = \frac{B_{\text{макс}}}{a + \Delta a},$$

where Δa - part of the aperture, diverted under induced oscillations, regions with the large nonlinearity of field and the thickness of aperture tube. Let us introduce for the simplification the equivalent value of the maximum induction

$$B'_{\text{макс}} = \frac{B_{\text{макс}}}{1 + \frac{\Delta a}{a}}.$$

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Then

$$G_{\text{макс}} = \frac{B'_{\text{макс}}}{a}.$$

Ratio a/s can be represented in the form

$$\frac{a}{S} = \frac{B'_{\text{макс}}}{S^2 G_{\text{макс}}} S. \qquad (3.133)$$

If assigned the parameters μ_0, D/S and the factor of defocusing γ_0, then is fixed/recorded the hardness of the lens

$$K^2 = S^2 \left(\frac{D}{S}\right)^2 \frac{eG}{p}.$$

As is noted in S 2.6, the necessary gradient of focusing field weakly depends on ratio D/S, somewhat increasing with the decrease of this

μ_0, γ_s, β
relation. With selection Λ is actually fixed/recorded product S^2G. Consequently, with given ones μ_0, γ_s and energy of injection all values in expression (3.133) are fixed/recorded except the period of focusing field. Thus,

$$\frac{a}{S} \div S.$$

Hence it follows that from the point of view of the guarantee of the permissible induction in the poles of lens with an increase in the period of focusing field S it is possible to increase the aperture of channel proportionally S^2. In this case the limiting current also increases proportionally S^2.

However, in the linear accelerators increase S is limited by the allowed values of the factor of defocusing and, therefore, possibly only with a simultaneous increase in the wavelength of accelerating field. Actually/really, from the expression for the factor of the defocusing

$$\gamma_s = \pi^2 \left(\frac{S}{\beta\lambda}\right)^2 \left(\frac{\Omega}{\omega}\right)^2$$

it follows that at the fixed values of specific acceleration and energy of injection the period of focusing field can be raised only proportional to the wavelength of accelerating field $S = \lambda$, and if is fixed/recorded the amplitude of middle field, then $S = \sqrt{\lambda}$.

If the aperture of channel is increased more rapidly than the wavelength, then, as it is noted, the factor of transit time in the accelerating system with drift tubes deteriorates. However, within considerable limits this increase of the aperture is possible that it makes it possible to raise the intensity of beam in the linear accelerators due to an increase in the sizes/dimensions of accelerator.

At the selected optimum value μ_0 the necessary value of gradient $G_{макс}$ can be lowered, and aperture is respectively increased due to the decrease of the factor of defocusing γ_s.

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The latter is achieved by the decrease of the absolute value of synchronous phase. However, this it conducts to the decrease of the capture region of particles into acceleration mode and, therefore, to the decrease of average/mean beam current.

Under $W_\lambda = \text{const}$ and other equal conditions an increase in the wavelength of accelerating field from 1.5 to 2 m raises limiting current 1.8 times. If the transverse phase volume of beam proves to be comparable with the channel capacity, then maximally possible current (3.129) increases still more significant, since an increase λ at given value ν_0^0 makes it possible to substantially raise channel capacity. Actually/really, according to expression (2.126),

$$V_K = \gamma v_\Phi^0 \frac{\beta\lambda}{S}\left(\frac{a}{\lambda}\right)^2 \lambda,$$

so that channel capacity is proportional to λ^3.

For the numerical estimations of a maximally attainable intensity of the accelerated beam it is necessary to have data about the phase volumes of the beams, injected into the linear accelerator. Are at the present time measured the phase volumes of proton beams for some ionic sources. Unfortunately, the results of measurements, obtained by the different authors, is difficultly to reliably compare, since processing experimental data was performed employing different procedures. For example, in work [95] is introduced a priori assumption about the axial symmetry of beam. In this work was determined the integral current distribution of beam by four-dimensional phase volume, i.e., the dependence of the value of four-dimensional volume from the portion of the full current, included in this part of the volume. In work [61] it is a priori assumed, that the sections of four-dimensional volumes by hyperplane y=y'=0 coincide with the projections of these volumes on the phase plane x, x', and was determined integral distribution on plane x, x' not of the full current of beam, but the current of particles, isolated with slots y=0, y'=0.

From the results of work [95] it follows that in the center section of the four-dimensional phase volume of the beam of protons, generated by duoplasmatron [2, 62, 104-112], the phase density is distributed evenly. According to the data of this work the value of four-dimensional phase density in the center section of the volume does not depend on the mode/conditions of ionic source. On the periphery of four-dimensional volume phase density decreases up to zero. Apparently, the irising of intense beam would make it possible to obtain beam with the highest ratio of full current to the four-dimensional phase volume.

According to the data of the authors, who measured the phase volumes of the proton beams, which contained from 90 to 99o/o of full current, the phase current density, defined as the ratio of full current to the two-dimensional projection of phase volume, proves to be lying/horizontal within comparatively narrow limits.

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Experimental data testify, apparently, about the fact that the phase current density weakly depends on the mode/conditions of source and over a wide range of a change in the beam current remains approximately constant. According to the data of work [93], phase current density in the beam, generated by duoplasmatron and measured at the output of the high-voltage accelerating tube, at different modes/conditions of source lies/rests in the limits of 150-220 mA/cm•mrad with the beam current to 150 mA. Phase current density in the beam, generated by high-frequency source [2, 113-115], somewhat below. For the modified source of the type of duoplasmatron [112] work [95] gives the value of phase current density 300 mA/cm•mrad with the beam current to 480 mA.

Let us accept as the original value of phase current density at the output of the injector

$$j_0 = \frac{I}{V_n} = 300 \text{ mA/cm·mrad.}$$

This value can be before the entrance into the linear accelerator raised. In the beam of protons, which emerges from the electrostatic accelerator (injector of linear accelerator), the particle they are distributed evenly on the phases of radio frequency voltage of linear accelerator, and the scatter of particles on the longitudinal impulses/momenta/pulses is substantially less than the vertical spread/scope of separatrix. Between the injector and the linear accelerator it is possible to establish/install the buncher, intended for increasing the portion of particles, seized into acceleration mode [2, 116-118]. Klystron type buncher, usually utilized for the ionic accelerators, is the cavity resonator, loaded with two span half-tubes. High-frequency field in the clearance between half-tubes modulates the longitudinal velocity of particles. Let us assume that a phase difference between the field in the buncher and the field in the first resonator of linear accelerator is selected in such a way that the particle, proving to be synchronous in the accelerator, flies the clearance of buncher at the moment of the passage of the field through zero toward increase. The velocity of synchronous particle in the buncher will not change. Particles, which anticipate/lead synchronous in this period of radio frequency voltage, will be inhibited, and the particles, which move after the synchronous, will accelerate themselves. In drift space between the buncher and the linear accelerator occurs grouping of particles. The length of drift is selected so that up to the moment/torque of the entrance of synchronous particle into the linear accelerator of the phase of remaining particles they would be as near as possible to the synchronous. Thus, even prior to the beginning of acceleration beam is modulated on the density along the longitudinal axis. Clusters follow with frequency of accelerating voltage. Buncher transforms the phase volume of beam on the plane of longitudinal vibrations ψ, $g=\frac{p-p_s}{p_s}$: due to an increase in the scatter of particles in the longitudinal velocities it is reduced the scatter of the larger part of the particles on the phases of accelerating field so that the phases of the majority of the particles, injected into the accelerator, would prove to be within the capture region.

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The strain of phase volume on plane $\psi, g = \frac{p-p_s}{p_s}$ is shown in Fig. 3.18. Region 1 corresponds to the phase volume of beam on the entrance into the buncher. After sinusoidal modulation of the longitudinal velocities of particles in the clearance of buncher the phase volume is converted into region 2. Region 3 - the result of the strain of phase volume in drift space. The same figure gives the boundary of the region of the capture of particles to acceleration mode. The process of preliminary beam bunching leads to an increase in the phase density in the four-dimensional phase volume of transverse coordinates and impulses/momenta/pulses. Actually/really the buncher in the first approximation, does not affect the transversing speeds of particles; therefore the scatter of particles on the transversing speeds in drift space is not changed, but increases a number of particles in each cross section of beam in the region of clustering. This effect does not contradict Liouville's theorem. In the equations of motion, which correspond to buncher and to the section of drift under the usual simplifying assumptions the variable/alternat ng are divided. Therefore in accordance with Liouville's theorem ust be retained the value of phase volume for each of three phase planes. However, because of modulation of the longitudinal velocities the

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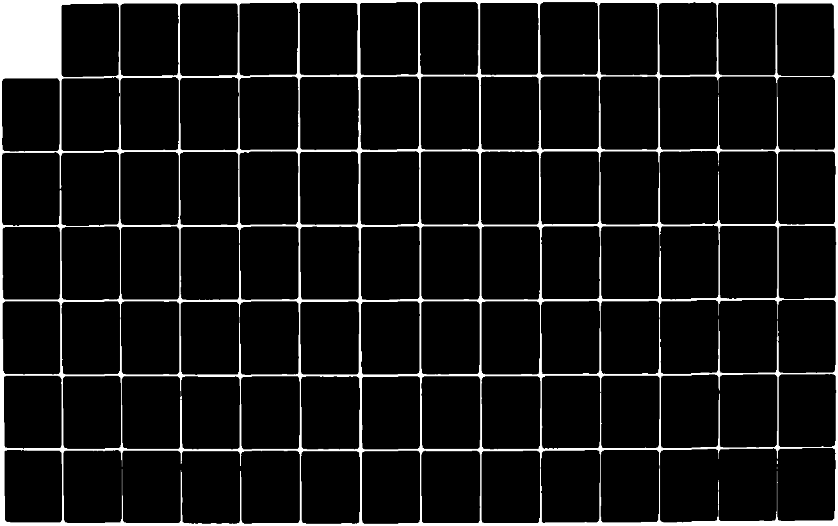

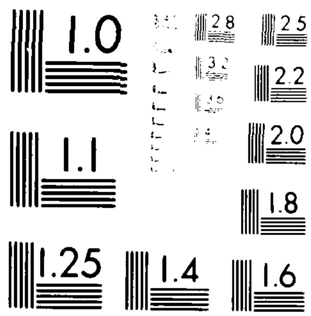

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condition of retaining/preserving/maintaining the number of particles in the element/cell of the four-dimensional volume of transverse coordinates and impulses/momenta/pulses is not satisfied, so that the corollary of Liouville about the retention/preservation/maintaining of phase density in this case does not occur.

At the absolute value of synchronous phase in limits of 30-40° buncher makes it possible to raise average/mean particle density in the clusters approximately/exemplarily two times [2]. Therefore during the estimation of the maximum peak current of clusters in the accelerator it is possible to be oriented toward the value of phase current density j=600 mA/cm·mrad. Since, judging according to preliminary experimental data, j=const, with an increase in the current of injection increases the transverse phase volume of beam. This leads to a reduction in the maximum permissible peak current.

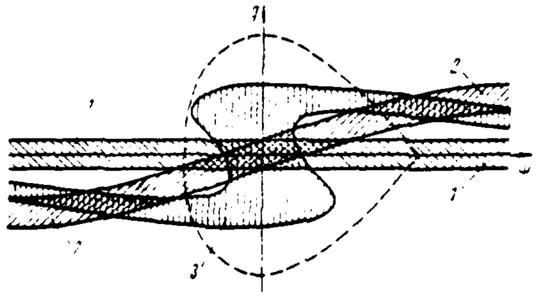

Fig. 3.18.

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Let us substitute the value of the transverse phase volume

$$V_п = \frac{I_{макс}}{j}$$

into formula (3.129). We will obtain the following dependence of a maximally possible peak current on the phase current density in the injected beam

$$I_{макс} = I_{пред}\frac{2h}{\sqrt{1+h^2}-h}. \qquad (3.134)$$

where

$$h = j\frac{S}{\mu_0\beta^2\gamma^2 I_0}. \qquad (3.135)$$

Let us examine as an example the short-wave accelerator of protons with following parameters [14]:

$$\mu_0 = 1{,}05;\quad \nu_\Phi^0 = 0{,}50;\quad S = 2\beta\lambda.$$
$$\beta_0 = 0{,}04;\quad a = 0{,}9\ см;\quad \lambda = 200\ см.$$

According to expression (3.132), the limiting value of peak current, which can be achieved/reached in this accelerator when $\frac{V_п}{V_к} = 1$, is equal to $I_{\text{пред}} = 1.67$ a. However, to the value/significance indicated in the experimentally measured parameters of proton beams it is impossible thus far to approach. From formula (3.135) we have h=0.19. Hence $I_{\text{макс}} = 0.31\ I_{\text{пред}}$. A maximally possible peak current of cluster proves to be equal to $I_{\text{макс}} = 520$ mA. The average/mean current of the accelerated beam in accordance with relationship/ratio (4.58) when $\cos\varphi_s = 0.8$ is $I_{\text{ср}} \approx 100$ mA. The given estimations can prove to be somewhat optimistic, since they assume that the beam is sufficiently well matched with the channel.

Returning to the question about the optimum identification of the parameters of the strong-focusing system of linear accelerator, it is possible to make following conclusions.

1. Calculation of channel for stability is conducted without taking into account space charge of beam. For this calculation it is possible to utilize diagrams of Smith-Glyukstern or diagram on plane γ_s, $\cos\mu_s$. In special cases can prove to be more convenient and other diagrams.

2. Should be designed channel for maximally large capacity. For this it is necessary to obtain the largest possible value/significance of the minimum frequency of transverse vibrations $\omega^{\prime}_{\min}$ of synchronous particle and the widest possible aperture. For

obtaining maximally possible value ω'_{max} it is necessary to select the optimum value of $\cos\mu_s$ of synchronous particle, ensuring greatest value/significance ν_Φ'' with the sufficiently low values of the gradient of focusing field, and the smallest possible value of ratio $S/\beta\lambda$ with that fixed/recorded ν_Φ''.

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The expansion of the aperture of the focusing channel, diverted under the free oscillations, is achieved:

1) by an increase in the wavelength of accelerating field, if aperture is limited by the value of magnetic induction in the poles:

2) by transition to the pulse supply of magnetic lenses (in the pulsed accelerators) if constraint of aperture is associated with the dissipation of power in the lenses;

3) with a reduction in the nonlinearity of focusing fields due to an improvement in the form of poles;

4) by the careful adjustment of the focusing lenses.

3. Calculation of structural/design and electrical allowances

for focusing channel should be conducted taking into account space charge of beam.

4. Matched initial conditions for envelope of particles and parameters of matching network also must be determined taking into account space charge of beam.

5. Ionic sources and optic/optics of injector must be selected of condition of obtaining highest possible phase current density. In the optimum case the transverse phase volume of beam must be approximately/exemplarily three times less than the capacity of the focusing channel of accelerator.

The majorities of working and planned proton linear strong-focusings accelrator serve as the injectors of proton synchrotrons. Therefore it is of interest at least under the simplest assumptions to rate/estimate maximally necessary current of injection in the proton synchrotron. Let us examine the strong-focusing proton synchrotron for which it is possible to utilize some relationships/ratios, meridional above.

In the circular accelerators any disturbances/perturbations of those focusing, fields periodically are repeated with the frequency, equal to the fundamental frequency of the rotation of particles. Let

Q - number of transverse vibrations, which fall for one revolution. With Q=1 the frequency of transverse vibrations coincides with the frequency of disturbances/perturbations and is excited the first external resonance, which virtually leads to the loss of beam. Remaining external resonances appear on the harmonics of fundamental frequency with

Q=n where n - the integers. With whole and half integral Q are excited parametric resonances. The first parametric resonance begins, when the frequency of the disturbances/perturbations doubly higher than frequency of transverse vibrations (Q=1/2). Remaining parametric resonances are excited with Q=n, n+1/2; n≥1. Value Q is selected between two forbidden values [119, 120]:

$$Q \approx n \pm 1/4.$$

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By selection Q is determined phase change of transverse vibrations in one period of the focusing structure

$$\mu = \frac{2\pi Q}{M},$$

where M - number of periods of the focusing structure at the orbit circumference. With the essential space charge of beam the frequencies of the incoherent of particles and oscillations of beam as whole diverge. The transverse sizes/dimensions of beam in the annular chamber of the synchrotron is considerably smaller than the transverse sizes/dimensions of chamber/camera; therefore phase change of coherent oscillations it remains close to μ_0. For the incoherent oscillations phase change will be mixed to the value, determined in the smooth approximation/approach by relationship/ratio (3.39). As a result of the smallness of the permissible frequency switch of incoherent oscillations the Coulomb parameter of beam is also low. With h<<1 for the incoherent oscillations we have

$$\mu = \mu_0 (1 - h).$$

Since each of the frequencies must not coincide with any resonance value/significance, this sets limitation on difference in both the frequencies

$$\Delta Q < \frac{1}{4},$$

or

$$\Delta\mu < \frac{\pi}{2M}.$$

Let us lead entire permissible frequency range for the displacement, connected with the space charge. Then the maximum value/significance of the Coulomb parameter is equal

$$h_{макс} = \frac{\pi}{2M\mu_0}.$$

But, on the other hand, from expression (3.36) it follows

$$I_{макс} = \frac{I_0}{c} \beta\gamma^2 \Omega_r^0 V_n h_{макс}.$$

In the circular accelerator, according to expression (2.106),

$$\Omega_r^0 = \mu_0 \frac{\beta c M}{L},$$

where L=MS - length of equilibrium orbit. Hence

$$I_{макс} = \frac{\pi\beta^2\gamma^2}{2L} V_n I_0.$$

Let us note that a maximally possible current of the particles injected into the circular accelerator is directly proportional to the transverse phase volume of beam, while in the linear accelerator maximum current decreases with an increase in the phase volume. This difference is connected with the different nature of the superimposed limitations: in the circular accelerators the limitation is superimposed on the frequency switch of transverse vibrations, and in the linear accelerators - to the transverse sizes/dimensions of beam.

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If the phase volume of the injected beam is lower than the capacity of annular chamber, then it is possible to increase beam current, utilizing multiple injection [121]. In the principle of increasing the current of the accelerated beam it is possible until the transverse phase volume of beam proves to be equal to the capacity of chamber/camera according to each degree of freedom. In this case attains the limiting value of the peak current

$$I_{\text{пред}} = \frac{\pi\beta^2\gamma^2}{2L} V_{\text{к}} I_0. \qquad (3.136)$$

Expression (3.136) is inconvenient for the estimations, since in the circular accelerators of transmission ability depends on energy of particles. If we want to compare the limiting values of current with different energies of injection, then into formula (3.136) one should introduce the value of the acceptance of annular chamber, since in the cyclic accelerators the acceptance on energy of particles does not depend. Utilizing relationship/ratio (2.127), we will obtain

$$I_{\text{пред}} = \frac{\pi\beta^3\gamma^3}{2L} A I_0. \qquad (3.137)$$

With the aid of formula (3.137) it is possible to rate/estimate the current which must be had at the output of the injector of the strong-focusing proton synchrotron. Between the linear accelerator-injector and the circular accelerator is established/installed high-frequency device/equipment debuncher), that works in the frequency of accelerating field of injector. Action of debuncher conversely to the action of buncher. Debuncher decreases the scatter of particles on the longitudinal impulses/momenta/pulses and smooths modulation of current density along the longitudinal axis of bundle. After debuncher the current of injection is close to constant, equal average current in the linear accelerator. The average/mean current of linear accelerator proves to be then peak current in the circular accelerator (if in the latter is not conducted preliminary bunching of beam in annular chamber). Therefore formula (3.137) gives estimation for the average/mean current of injector. It is necessary to consider that the part of the current of injection (to 50o/o) can be lost because the scatter of particles on the longitudinal impulses/momenta/pulses after debuncher exceeds the boundaries, adjusted by capture region. In connection with this the average/mean current of linear accelerator must be approximately/exemplarily two times of more than the value, determined by relationship/ratio (3.137),

$$I_{\text{ср. инж}} = \frac{\pi\beta^2\gamma^3}{L} A I_0. \tag{3.138}$$

Let us determine on the basis of formula (3.137) a maximally possible number of particles, accelerated in one cycle of acceleration by the strong-focusing proton synchrotron, $N_{\text{макс}}$.

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The average/mean current of particles in the annular chamber

$$I_{cp} = \frac{eNv_s}{L},$$

where N - number of particles in the chamber/camera. As it will be shown below [see expression (4.58)], with the complete filling of phase stability region average/mean beam current is connected with the peak current of cluster with the equality

$$I_{cp} = \frac{|\varphi_s|}{\pi} I.$$

Synchronous phase is counted off from the moment/torque when field reaches maximum. From the latter/last two relationships/ratios it follows

$$N_{макс} = \frac{\varphi_s}{\pi e \beta c} L I_{пред}.$$

After substituting the limiting value of peak current (3.137), and taking into account expression (3.22), we will obtain

$$N_{макс} = \frac{A|\varphi_s|}{2r_p} \beta^2 \gamma^3. \qquad (3.139)$$

Value

$$r_p = \frac{e^2}{4\pi\varepsilon_0 m_0 c^2}$$

this is a "classical radius" of the proton: $r_p = 1{,}54 \cdot 10^{-16}$ см.

Energy of injection in strong-focusing proton synchrotron to the exit energy 60-70 GeV [122] is equal to W_s=100 MeV; orbit circumference L=1.48 km; synchronous phase $\varphi_s = 60$; the acceptance of accelerator A=2.5 cm·mrad, which corresponds to the amplitude of free oscillations a=3 cm (sizes/dimensions of chamber/camera 20x12 cm). Substituting the appropriate values into formula (3.138), we obtain $I_{ср.инж} \approx 180$ mA. It is assumed that half of this current will be seized into acceleration mode. Maximum peak current of clusters in annular chamber $I_{пред} \approx 90$ mA. A maximum number of protons in one cycle of acceleration is equal, according to (3.139), $N_{макс} \approx 2 \cdot 10^{12}$.

Linear accelerator-injectors for the strong-focusing proton synchrotrons have, as a rule, wavelength of accelerating field λ=1.5-2 m. The comparison of the current of injection, which ensures complete filling of annular chamber, with the maximum value of average/mean beam current in the short-wave linear accelerator, obtained above, shows that a maximally possible current of linear accelerator is found at best on the limit of necessary. Thus, at present precisely injection limits a maximum number of particles in the cycle of proton synchrotron.

S 3.8. Space-charge effect on the beam focusing in the longitudinal magnetic field.

Let us examine beam focusing with the high phase current density in the longitudinal magnetic field. For the beams with zero phase volume a similar task repeatedly was set forth in the literature [63, 64, 85, 123-125]. Let us examine below beams whose phase volume in the general case is different from zero.

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For reasons, given in § 3.1, we will consider beam in the linear accelerator continuous. The current of steady beam corresponds to the peak current of clusters. Further, let us suppose that the proper field of beam is formed only by the accelerated particles. The calculation procedure remains the same, as in the case of the strong-focusing fields; however, calculation is simplified, since for the equations of motion of particles in the stationary longitudinal magnetic field it is easy to indicate the first integrals, without resorting to any supplementary assumptions. In particular, in longitudinal magnetic field one of constants of motion - Hamiltonian of particle.

The components of the proper field of beam are obtained above and are determined by expressions (2.45). Let us substitute the components of proper field in vector equation (2.5). Instead of equations (2.295) we will obtain the equations of motion, which consider the space charge of the beam

$$\begin{aligned} \frac{d^2x}{dt^2} &= 2\omega_L \frac{dy}{dt} + \frac{1}{2}\Omega_\varphi^2 x - \frac{e}{m_0\gamma^3}\cdot\frac{\partial U}{\partial x}; \\ \frac{d^2y}{dt^2} &= -2\omega_L \frac{dx}{dt} + \frac{1}{2}\Omega_\varphi^2 y - \frac{e}{m_0\gamma^3}\cdot\frac{\partial U}{\partial y}. \end{aligned} \tag{3.140}$$

In equations (3.140) function U(x, y, z) - the potential of proper field. Frequencies ω_L and Ω_q are determined respectively by equalities (2.294), (2.293). Let into the focusing channel enter the beam of round cross section. In view of the axial symmetry of all applied fields the beam and subsequently will remain axisymmetric. In this case it is assumed that also the four-dimensional phase volume of beam possesses axial symmetry. The potential of proper field depends in the transverse plane only on a radius - vector r. After passing in equations (3.140) to the polar coordinates, in exchange for (2.296) we will obtain

$$\begin{aligned} \frac{d^2r}{dt^2} &= r\left(\frac{d\psi}{dt}\right)^2 - 2\omega_L r\frac{d\psi}{dt} + \frac{1}{2}\Omega_q^2 r - \frac{e}{m_0\gamma^3}\cdot\frac{\partial U}{\partial r}; \\ r\frac{d^2\psi}{dt^2} &= -2\omega_L\frac{dr}{dt} - 2\frac{dr}{dt}\cdot\frac{d\psi}{dt}. \end{aligned} \qquad (3.141)$$

Into second equation (3.141) the potential of the proper field of beam does not enter. Consequently, constant of motion (2.297) is retained, so that as before the angular rate of rotation of particles relative to the axis of accelerator is determined by the equality

$$\frac{d\psi}{dt} = -\omega_L + \frac{M}{r^2}. \qquad (3.142)$$

The substitution of angular velocity (3.142) into first equation (3.141) gives

$$\frac{d^2r}{dt^2} + \omega_r^{0\,2} r - \frac{M^2}{r^3} + \frac{e}{m_0\gamma^3}\cdot\frac{\partial U}{\partial r} = 0. \qquad (3.143)$$

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Parameter ω_r^0 is assigned by expression (2.301)

$$\omega_r^0 = \sqrt{\Omega_L^2 - \frac{1}{2}\Omega_q^2}$$

and it is the frequency of a radius - the vector of the particle, which did not have initial rotation, in the beam of zero intensity. Let us examine the matched beams, which retain a constant radius of section along the axis of channel. Then the potential of the proper field of beam - function only from a radius - vector r. After multiplying equation (3.143) on dr/dt and after integrating, we will obtain integral of motion, analogous (2.306):

$$\mathrm{I} = \left(\frac{dr}{dt}\right)^2 - \frac{M^2}{r^2} - \omega^{02} r^2 - \frac{2e}{m_0\gamma^3} U(r). \qquad (3.144)$$

The sum of first two terms in the right side of expression (3.144) is, as shown in S 2.9, the square of the disordered component of complete linear velocity in the plane, perpendicular to the axis of the bundle:

$$v_\perp^2 = \dot{r}^2 - \frac{M^2}{r^2}.$$

Therefore constant of motion (3.144) can be represented in the form

$$\mathrm{I}(r, v_\perp) = v_\perp^2 - \omega^{02} r^2 - \frac{2e}{m_0\gamma^3} U(r). \qquad (3.145)$$

Let us restrict the circle of the stationary distributions in question by the case

$$n(r, v_\perp) = f(\mathrm{I}). \qquad (3.146)$$

Let us note that constant of motion $\mathrm{I}(r, v_\perp)$ - the linear combination of two independent integrals

$$\mathrm{I} = 2(H - \omega_L M). \qquad (3.147)$$

where H - Hamiltonian of particle, and M - generalized moment of momentum (2.297). Actually/really, equation (3.140) it is possible to lead to the canonical form, if we introduce the generalized momenta, canonical-conjugated/combined with the Cartesian coordinates

$$P_x = \dot{x} - \omega_L y; \quad P_y = \dot{y} - \omega_L x.$$

Hamiltonian, as it is possible directly to be convinced, takes the form

$$H = \frac{1}{2}\left[(P_x - \omega_L y)^2 + (P_y - \omega_L x)^2\right] - \frac{\Omega_0^2}{4}(x^2 + y^2) + \frac{e}{m_0\gamma^3} U(x, y).$$

and constant of motion (2.297)

$$M = xP_y - yP_x$$

regarding is the generalized moment of momentum [21]

$$\mathbf{M} = [\mathbf{r}, \mathbf{P}].$$

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It is easy to show that linear combination (3.147) gives expression (3.144). Thus, in the case (3.146) phase density is actually the function of two independent constants of motion H and M.

Further, let us suppose that the stationary distribution of phase density corresponds to the microcanonical

$$n(r, v_\perp) = n_0 \delta\left[\mathrm{I}(r, v_\perp) - \mathrm{I}_0\right].$$

Density distribution of space charge according to the beam section

$$\varrho(r) = 2\pi e n_0 \int_0^\infty \delta\left[\mathrm{I}(r, v_\perp) - \mathrm{I}_0\right] v_\perp \, dv_\perp. \tag{3.148}$$

This distribution must satisfy the integral condition

$$I = 2\pi v_s \int_0^{R_c} \varrho(r)\, r\, dr = \mathrm{const}. \tag{3.149}$$

Let us substitute into integral (3.148) function (3.145) and will produce integration by the replacement of variable/alternating analogously with § 3.2. Taking into account (3.149) we will obtain

$$\varrho(r) = \frac{I}{\pi v_s R_c^2} = \text{const}.$$

Function U(r) - the internal potential of the evenly charged/loaded circular cylinder

$$\frac{d^2U}{dr^2} + \frac{1}{2}\frac{dU}{dr} = -\frac{1}{\varepsilon_0}\varrho.$$

Under boundary conditions U(0)=dU/dr(0)=0 we have

$$U(r) = -\frac{I}{4\pi\varepsilon_0 v_s}\left(\frac{r}{R_c}\right)^2.$$

Constant of motion (3.144) can be now represented in the form

$$\left(\frac{dr}{dt}\right)^2 + \frac{M^2}{r^2} + \omega_r^2 r^2 = I_0.$$

Here

$$\omega_r^2 = \omega_r^{02} - \frac{2c^3 I}{\beta\gamma^3 I_0 R_c^2}, \qquad (3.150)$$

where I_0 - standard strength of current (3.22). According to the condition, all particles in the beam have one and the same value/significance of constant of motion $I=I_0$. The peripheries of beam $r=R_c$ reach the particles, which did not possess initial rotation relative to the axis of accelerator. Phase trajectory of these particles on plane r, $\dot{r}$

$$\dot{r}^2 - \omega_r^2 r^2 = \omega_r^2 R_c^2. \qquad (3.151)$$

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Trajectory (3.151) covers the representative points of all particles with M≠0 (see Fig. 2.26). According to expression (3.151), ω_r - the frequency of a radius - the vector of the particle for which M=0, in the beam with the nonzero intensity. As shown in S 2.9, ellipse (3.151) coincides with the projection of four-dimensional phase volume on plane x, $\dot{x}$. The transverse phase volume of beam, regarding (2.2), is equal to projected area on plane x, $\dot{x}$. Thus, for the matched beam with a radius of R we have

$$V_\perp = \pi\omega_r R^2$$

A radius of matched beam with the same phase volume and zero intensity is determined by the equality

$$V_\parallel = \pi\omega_r^0 R_c^{02}$$

Hence

$$R_c = R_c^0 \sqrt{\frac{\omega_r^0}{\omega_r}}. \qquad (3.152)$$

If we introduce into equality (3.152) of expression for ω from formula (3.150) and to solve equation relative to R_c, then we will obtain

$$R_c^2 = \frac{c^2 I}{\omega_r^{02}\beta\gamma^3 I_0} + \sqrt{R_c^{04} + \left(\frac{c^2 I}{\omega_r^{02}\beta\gamma^3 I_0}\right)^2},$$

or

$$R_c = R_c^0 \sqrt{h + \sqrt{1 + h^2}}. \qquad (3.153)$$

The dimensionless combination of values

$$h = \frac{c}{\omega_r^0 \beta\gamma^2 I_0} \cdot \frac{I}{V_\parallel}$$

is the Coulomb parameter of beam in the longitudinal magnetic field. According to expressions (3.152), (3.153),

$$\omega_r = \omega_r^0 \left(\sqrt{1 + h^2} - h\right). \qquad (3.154)$$

Expressions for a radius of matched beam and frequency of a radius - the vector of particle formally coincide with the analogous relationships/ratios, obtained above in the smooth approximation/approach for the beam in the strong-focusing channel. The degenerate and canonical distributions also lead to with respect to the coinciding formulas. Hence it follows that ratios R_c R_c^0 and ω_c ω_c^0 in the longitudinal magnetic field also noncritically depend on the form of the stationary distribution of phase density.

The maximum permissible phase volume of beam with the nonzero intensity is connected with the channel capacity (2.316) with the equality, which directly escape/ensues from relationship/ratio (3.154)

$$V_{\kappa h}=V_{\kappa}\left(\sqrt{1+h^2}-h\right).$$

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Maximally possible peak beam current in the channel with the longitudinal magnetic field

$$I_{\text{макс}}=I_{\text{пред}}\left[1-\left(\frac{V_{\text{п}}}{V_{\text{к}}}\right)^2\right],$$

where

$$I_{\text{пред}}=\frac{1}{2c}\,\omega_r^0\beta\gamma^2 I_0 V_{\kappa}.$$

After substituting expression (2.316), we will obtain

$$I_{\text{пред}}=2\pi^2\left(\frac{\omega_r^0}{\omega}\right)^2\left(\frac{a}{\lambda}\right)^2\beta\gamma^3 I_0. \qquad (3.155)$$

If we for the comparison rewrite formula for the limiting current in strong-focusing channel (3.132) in the form

$$I_{\text{пред}}=2\pi^2\left(\frac{\sqrt{\omega_{r\,\text{чнп}}^0\Omega_r^0}}{\omega}\right)^2\left(\frac{a}{\lambda}\right)^2\beta\gamma^3 I_0, \qquad (3.156)$$

then it is possible to see that both expressions are close, only in formula (3.156) instead of frequency ω^0 stands square root from the product of the medium frequency of transverse vibrations for the minimum instantaneous frequency. Let us compare formulas (3.155) and (3.156) at the identical values of a/λ and $\beta\gamma^3$. Let us accept, according to expression (2.320),

$$\omega_r^{0\,2} = \frac{1}{2}\,\Omega^2,$$

where Ω - initial frequency of small longitudinal vibrations. For the linear proton accelerator I-100 [14]

$$\mu_0 = 1{,}05;\quad \nu_\Phi^0 = 0{,}50;\quad S = 2\beta\lambda;\quad \left(\frac{\Omega}{\omega}\right)_0^2 = 8{,}73\cdot 10^{-3}.$$

Hence

$$\frac{\omega_r^0{}_{\text{мин}}\Omega^0}{\omega^2} = 3{,}3\cdot 10^{-3};\quad \left(\frac{\omega_r^0}{\omega}\right)^2 = 4{,}4\cdot 10^{-3}.$$

Thus, in the longitudinal magnetic field the limiting current of beam proves to be approximately/exemplarily to 30o/o higher than in the strong-focusing channel. This is connected with the low value/significance of minimum instantaneous frequency in the channel of quadrupole lenses, i.e., with the presence relative to deep modulation of envelope, which makes it necessary to decrease the mean radius of beam in comparison with the assigned aperture. Thus, in the example

$$\frac{\Omega_r^0}{\omega} = 8{,}3\cdot 10^{-2};\qquad \frac{\omega_r^0{}_{\text{мин}}}{\omega} = 4{,}0\cdot 10^{-2}.$$

in question. In the channel with the longitudinal magnetic field

$$\frac{\omega_r^0}{\omega} = 6{,}6\cdot 10^{-2}.$$

The mean radius of beam in the strong-focusing channel is lower than the radius of beam in the channel with the longitudinal magnetic field, but maximum size proves to be more.

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Chapter 4.

Longitudinal vibrations of particles in beams with the high density of space charge.

4.1. Potential distribution of self-congruent field along the longitudinal axis of cluster.

During the analysis of transverse vibrations of particles in the beams with the high density of space charge we disregarded/neglected the dependence of the transverse components of the proper field of beam on the longitudinal coordinate. This made it possible to avoid the difficulties, connected with the examination of complete task in the six-dimensional phase space, and to bring together the latter to the determination of the self-consistent solutions for the proper field of beam and particle trajectories taking into account the distribution of phase density in the four-dimensional space of transverse coordinates and impulses/momenta/pulses. However, it is important to examine the longitudinal vibrations of particles in the intense beams, since previously it cannot be indicated, what effect more greatly limits the limiting current of beam - transverse or longitudinal Coulomb pushing apart. As it will be shown, response/answer to this question depends on the identification of the parameters of accelerator.

The longitudinal forces of electrostatic pushing apart susceptibly/critically depend on the length of clusters and, therefore, they must be sensitive to the law of density distribution of space charge along the longitudinal axis of cluster. The correct formulation of the problem requires the determination of the self-consistent potential distribution within the cluster. To avoid the formulation of the problem is fully possible only in such a case, when we disregard/neglect the dependence of the longitudinal component of the proper field of beam on the transverse coordinates.

In § 3.1 based on the example of the general ellipsoid it was shown that the longitudinal component of the proper field of space charge very weakly depends on the relation of the transverse semi-axes (see Fig. 3.1). The longitudinal component of field has the greatest value/significance with the circular cross section of cluster.

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Further as in Chapter 1, let us suppose that the amplitude of the longitudinal component of accelerating field does not depend on transverse coordinates and is only the function of the longitudinal coordinate

$$E = E(z).$$

Since the longitudinal component of proper field, by hypothesis, on transverse coordinates does not depend, density distribution of charge along the longitudinal axis must depend only on the longitudinal coordinate

$$\varrho = \varrho(z).$$

Thus, in each cross section of beam the density of space charge is constant, which is in accordance with the assumption about the stationary microcanonical distribution of phase density in the four-dimensional phase space of transverse coordinates and impulses/momenta/pulses. The assumptions indicated in principle simplify the task about the phase stability and they make it possible to bring together it to the two-dimensional on the phase plane of longitudinal coordinates and impulses/momenta/pulses.

Let us introduce also the following assumptions, which no longer carry fundamental character, but the simplifying calculations:

1. The effect of the metallic walls of channel on the proper field of beam is negligibly small.

2. Bunches of particles are well formed, and all particles, not seized into acceleration mode, fell out from beam. Let us note that the task about the particles out of the separatrix is substantially nonlinear. The deletion of such particles leads to the overestimation of the effect of longitudinal pushing apart.

3. Particles are accelerated in field of traveling wave. For the discrete/digital accelerating structures such field is the equivalent traveling wave.

Let φ_s - synchronous phase in the absence of longitudinal Coulomb pushing apart; z_s - longitudinal coordinate of synchronous particle in the assigned applied fields in the beam of zero intensity. In other words, z_s - moving coordinate of point at the front of the traveling wave in which would be located synchronous particle, if the forces of Coulomb pushing apart were absent. Let us introduce the moving coordinate system, rigidly connected with the front of the traveling wave. Longitudinal coordinate in this system we will count off from point z_s

$$\zeta = z - z_s.$$

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Equation of motion for the arbitrary particle in the cluster takes the form

$$\frac{dW}{dz} = eE\cos\varphi + eE_{\kappa}(\zeta), \qquad (4.1)$$

where

$$\varphi = \omega\left(t - \int_0^z \frac{dz}{v_s}\right)$$

there is a phase of the traveling wave in which is located the particle; E - amplitude of the traveling wave; E_{κ} - longitudinal component of the proper field of cluster. The current velocity of synchronous particle regarding is equal at each moment of the time of the velocity of the motion of the front of the traveling wave. Therefore an increase in energy of synchronous particle per unit of length does not depend on the forces of the Coulomb pushing apart:

$$\frac{dW_s}{dz} = eE\cos\varphi_s. \qquad (4.2)$$

Let φ_s^* - new value/significance of synchronous phase, shifted relatively φ_s due to the action of Coulomb forces; ζ - distance from the synchronous particle to the instantaneous value of coordinate z_s. Then

$$eE\cos\varphi_s = eE\cos\varphi_s^* + eE_{\kappa}(\zeta_s).$$

Let us introduce into the examination canonical-conjugated/combined variable/alternating, used in Chapter I: phase of particle ψ (1.47), calculated off the synchronous phase in the field without the space charge, and energy difference of synchronous and nonsynchronous particles (1.48)

$$\psi = \varphi - \varphi_s;$$
$$p_{\psi} = W_s - W.$$

After deducting equation (4.1) from equation (4.2), we obtain

$$\frac{dp_{\psi}}{dz} = eE\left[\cos\varphi_s - \cos(\psi + \varphi_s)\right] - eE_{\kappa}(\zeta). \qquad (4.3)$$

Second equation, which describes longitudinal vibrations, (1.51) - purely kinematic. The form of this equation upon consideration of the forces of Coulomb pushing apart is not changed. Let us pass in equation (4.3) to independent by the variable/alternating t. As a result we will obtain the following system di first-order equations

$$\frac{d\psi}{dt} = \frac{\omega}{\gamma^2 m_s v_s} p_\psi;$$
$$\frac{dp_\psi}{dt} = ev_s E\left[\cos\varphi_s - \cos(\psi - \varphi_s)\right] - ev_s E_\kappa(\zeta). \qquad (4.4)$$

Equations (4.4) are written in the laboratory coordinate system. Therefore, according to expression (2.45),

$$E_\kappa(\zeta) = -\frac{1}{\gamma^2} \cdot \frac{\partial U}{\partial \zeta}(x, y, \zeta). \qquad (4.5)$$

where U - potential of the proper field of cluster in the laboratory system. Since we disregarded/neglected the dependence of the longitudinal component of the proper field of cluster on the transverse coordinates, we can replace partial derivative in equality (4.5) of the total derivative

$$E_\kappa(\zeta) = -\frac{1}{\gamma^2} \cdot \frac{d}{d\zeta} U(\zeta).$$

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Differentials from the phase of particle and from the longitudinal coordinate are connected with the relationship/ratio

$$d\psi = -\frac{\omega}{v_s} d\zeta.$$

Hence

$$E_\kappa(\psi) = \frac{\omega}{\gamma^2 v_s} \cdot \frac{dU}{d\psi}(\psi).$$

Equations (4.4) can be now represented in canonical form

$$\frac{d\psi}{dt}=\frac{\omega}{\gamma^2 p_s v_s}p_\psi;$$
$$\frac{dp_\psi}{dt}=ev_sE\left[\cos\varphi_s-\cos(\psi-\varphi_s)\right]-\frac{e\omega}{\gamma^2}\cdot\frac{dU}{d\psi}(\psi). \quad (4.6)$$

The Hamiltonian of particle takes the form

$$H(\psi,p_\psi,t)=\frac{\omega}{2\gamma^2 p_s v_s}p_\psi^2-ev_sE\left[\sin(\psi-\varphi_s)-\right.$$
$$\left.-\psi\cos\varphi_s\right]+\frac{e\omega}{\gamma^2}U(\psi). \quad (4.7)$$

Expression (4.7) is the generalization of Hamiltonian (1.53) to the case when it is not possible to disregard the effect of the proper field of beam to the longitudinal vibrations of particles.

Let us further examine only conservative approximation/approach, by proposing as in Chapter 1 that Hamiltonian (4.7) - constant of motion. For simplification in further recordings it is convenient to introduce instead of electric potential $U(\psi)$ the new potential function

$$\Phi(\psi)=\frac{\psi\cos\varphi_s-\sin(\psi-\varphi_s)}{\sin\varphi_s}+\frac{e}{p_s v_s\gamma}\left(\frac{\omega}{\Omega_c}\right)^2U(\psi).$$

Taking into account equality (1.70) Hamiltonian (4.7) can be then converted as follows

$$H_1(\psi,p_\psi)=\frac{\omega}{2\gamma^2 p_s v_s}p_\psi^2+\frac{\omega p_s v_s}{\gamma}\left(\frac{\Omega_c}{\omega}\right)^2\Phi(\psi).$$

Equation of phase trajectory on plane ψ, p_ψ, corresponding to the given value of H=const,

$$p_\psi=\pm\sqrt{2\gamma}\,p_s v_s\frac{\Omega_c}{\omega}\left[\frac{\omega\gamma}{p_s v_s\Omega_c^2}H-\Phi(\psi)\right]^{1/2}.$$

Let H_c - value of Hamiltonian, which corresponds to separatrix. With the designation

$$\frac{\omega\gamma H_c}{p_s v_s \Omega_c^2} = \Phi_c \qquad (4.8)$$

the equation of separatrix takes the form

$$P_{\psi c}(\psi) = \pm\sqrt{2\gamma}\, p_s v_s \frac{\Omega_c}{\omega}\sqrt{\Phi_c - \Phi(\psi)}. \qquad (4.9)$$

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In the equation of separatrix (4.9) are unknown function $\Phi(\psi)$ and number Φ_c, which are subject to determination with the assigned peak current of cluster. If $\Phi(\psi)$ and Φ_c will be determined, then by this very we will completely calculate phase stability region. The coordinate of a singular point of the type of center on the phase plane corresponds to the minimum of potential function [23, 24]

$$\frac{d\Phi}{d\psi}(\Psi_0) = 0; \frac{d^2\Phi}{d\psi^2}(\psi_0) > 0. \qquad (4.10)$$

Phase of synchronous particle in the intense beam

$$\varphi_s^* = \varphi_s - \psi_0.$$

A singular point of the type of saddle corresponds to the maximum of the potential function

$$\frac{d\Phi}{d\psi}(\psi_c) = 0; \quad \frac{d^2\Phi}{d\psi^2}(\psi_c) < 0. \ (4.11)$$

Potential function and separatrix is schematically shown in Fig. 4.1. In contrast to Fig. 1.5 singular point of the type of center by Fig. 4.1 is displaced relative to the origin of coordinates. The boundaries of the region of phase stability ψ_c, ψ_κ (see Fig. 4.1) satisfy condition $p_{\psi c} = 0$, whence it follows

$$\Phi(\psi_c) = \Phi_c; \quad \Phi(\psi_\kappa) = \Phi_c. \qquad (4.12)$$

By equations (4.12) is determined parameter Φ_c and the left boundary of the region of phase stability ψ_κ.

In particular, if Coulomb forces are absent, then equations (4.10), (4.11), (4.12) lead to the results, obtained in § 1.2. In the beam of zero intensity $U(\psi)\equiv 0$ and $\Phi(\psi)=\Phi_0(\psi)$, where

$$\Phi_0(\psi)=\frac{\psi\cos\varphi_s-\sin(\psi-\varphi_s)}{\sin\varphi_s}. \qquad (4.13)$$

Hence

$$\psi_0=0;\quad \psi_c=-2\varphi_s;\quad \psi_\kappa\approx\varphi_s;$$
$$\Phi_c=\Phi_0(-2\varphi_s)=1-2\varphi_s\,\mathrm{ctg}\,\varphi_s.$$

In this case the equation of separatrix (4.9) coincides with equation (1.57).

Let $n(\psi, p_\psi)$ - function of the distribution of phase density on plane ψ, p_ψ.

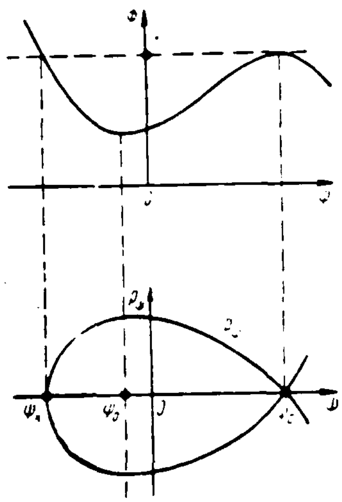

Fig. 4.1.

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During any stationary distribution

$$n(\psi, p_\psi) = f(H)$$

the clusters do not fluctuate, since the lines of equal density coincide with the phase trajectories. According to the condition, out of the separatrix the representative points are absent. Following work [126], let us accept simple assumption relative to the law of the distribution of the representative points within the separatrix, namely let us assume that within the separatrix the phase density is constant. For the phase trajectories, arranged/located within the separatrix, $H < H_c$; for the trajectories, which pass out of the separatrix, $H > H_c$. Hence

$$\begin{aligned} n &= n_0 \quad \overset{(1)}{\text{при}} \quad H < H_c; \\ n &= 0 \quad \overset{(1)}{\text{при}} \quad H > H_c. \end{aligned} \qquad (4.14)$$

Key: (1). with.

Function (4.14) describes the two-dimensional distribution of phase density, analogous to singular distribution (3.41) in the four-dimensional space.

Let us compose difference $H(\psi, p_\psi) - H_c$. According to expressions (4.8), (4.9),

$$H(\psi, p_\psi) - H_c = \frac{\omega}{2\gamma^2 p_s v_s}\left[p_\psi^2 - p_{\psi c}^2(\psi)\right]. \qquad (4.15)$$

The density of space charge changes along the longitudinal axis of cluster according to the law

$$\varrho(\psi) = e\int_{-\infty}^{+\infty} n(\psi, p_\psi)\, dp_\psi.$$

Integrand exists $n=n_0$ when $p_\psi^2 < p_{\psi c}^2$ and $n=0$ when $p_\psi^2 > p_{\psi c}^2$. Thus,

$$\varrho(\psi) = en_0\int_{-p_{\psi c}}^{+p_{\psi c}} dp_\psi = 2en_0 p_{\psi c}(\psi). \qquad (4.16)$$

The law of density distribution of charge along the longitudinal axis of cluster repeats the course of separatrix. If I - peak beam current and R_c - the mean radius of cluster, then

$$\varrho_{\text{макс}} = \frac{I}{\pi v_s R_c^2}. \qquad (4.17)$$

After assuming in expression (4.16) $\psi = \psi_0$ and after equating to its expression (4.17), we will obtain

$$n_0=\frac{I}{2\pi v_e R_c^2 p_{\psi c}(\psi_0)}. \qquad (4.18)$$

Taking into account formulas (4.9), (4.18) equality (4.16) is reduced to the form

$$\varrho(\psi)=\frac{I}{\pi v_e R_c^2}\sqrt{\frac{\Phi_c-\Phi(\psi)}{\Phi_c-\Phi(\psi_0)}}. \qquad (4.19)$$

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By formula (4.19) the density of space charge $\varrho(\psi)$ is expressed as the potential of the proper field of beam. This, in the principle, it makes it possible to compose equation for the potential of self-congruent field. For the calculations it is more convenient instead of differential Poisson's equation to use the general/common/total integral form of the solution

$$U=\frac{1}{4\pi\varepsilon_0}\int_V \frac{\varrho dV}{R}, \qquad (4.20)$$

where V - volume, occupied with space charge; R - radius-vector in the three-dimensional space. Substituting in equality (4.20) for the potential of proper field its expression through function $\Phi(\psi)$ and instead of ρ - expression (4.19), we will obtain integral equation for function $\Phi(\psi)$. Since the amplitude of accelerating field does not depend on transverse coordinates, in the absence of space charge to all particles independent of their misalignment corresponds one and the same separatrix. All particles of cluster are arranged/located within the circular cylinder, limited by flat/plane ends/faces. The length of cylinder is determined by phase stability region overall for all particles. As it will be shown, capture region on the phases in the presence of Coulomb pushing apart is reduced. The longitudinal component of the field of the space charge, limited by the cylinder of the finite length l, is maximum on the axis and somewhat weakens to the periphery of cylinder (Fig. 4.2). Therefore phase stability region must be reduced for the axial particles more than for the peripheral ones. However, since we disregarded/neglected the dependence of the longitudinal component of the proper field of cluster on the transverse coordinates, one should consider that and in the presence of space charge the cluster is limited by circular flat-topped cylinder.

Subsequently let us examine the particles, which lie on the axis of bundle. Let us calculate potential on the axis of a cylindrical cluster, radius R_{r}. in the coordinate system, relative to which the cluster rests (Ts-system).

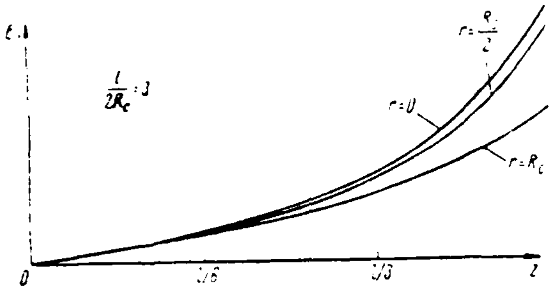

Fig. 4.2.

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Upon transfer from the laboratory coordinate system, rigidly connected with the accelerator, to ~~Ts~~ U-system longitudinal sizes/dimensions grow/rise γ once, so that in the same sense increases each element of volume and falls the density of the space charge

$$\varrho_{\text{п}}(\zeta_{\text{п}}) = \frac{1}{\gamma}\varrho(\zeta). \qquad (4.21)$$

Let $\zeta_{\text{с}}, \zeta_{\text{к}}$ - coordinate of beginning and end/lead of the cluster in the laboratory coordinate system. Coordinates of beginning and end/lead of the cluster in ~~Ts~~ U-system

$$\zeta_{\text{сп}} = \gamma\zeta_{\text{с}}; \quad \zeta_{\text{кп}} = \gamma\zeta_{\text{к}}.$$

Further, $r, \varphi, \xi_{\text{п}}$ - coordinate of the current point of integration in the cylindrical coordinate system, which accompanies cluster (Fig. 4.3). Distance from the current point of integration to the point on the axis with longitudinal coordinate $\zeta_{\text{п}}$ is equal

$$R_{\text{п}} = \sqrt{r^2 - (\zeta_{\text{п}} - \xi_{\text{п}})^2}.$$

Potential on the axis at point $\zeta_{\text{п}}$ is determined by the integral

$$U_{\text{п}}(\zeta_{\text{п}}) = \int\limits_{\zeta_{\text{сп}}}^{\zeta_{\text{кп}}} \int\limits_{0}^{R_c} \int\limits_{0}^{2\pi} \frac{\varrho_{\text{п}}(\xi_{\text{п}})\, r\,dr\,d\varphi\,d\xi_{\text{п}}}{4\pi\varepsilon_0 \sqrt{r^2 + (\zeta_{\text{п}} - \xi_{\text{п}})^2}}.$$

In each elementary disk (see Fig. 4.3) the density of space charge is distributed evenly. Therefore internal integrals can be elementarily calculated

$$U_{\text{п}}(\zeta_{\text{п}}) = \frac{1}{2\varepsilon_0} \int\limits_{\zeta_{\text{сп}}}^{\zeta_{\text{кп}}} \varrho_{\text{п}}(\xi_{\text{п}}) \left[\sqrt{R_c^2 + (\zeta_{\text{п}} - \xi_{\text{п}})^2} - \sqrt{(\zeta_{\text{п}} - \xi_{\text{п}})^2}\right] d\xi_{\text{п}}.$$

In the laboratory coordinate system we have

$$U(\zeta) = \gamma U_{\text{п}}(\zeta_{\text{п}}).$$

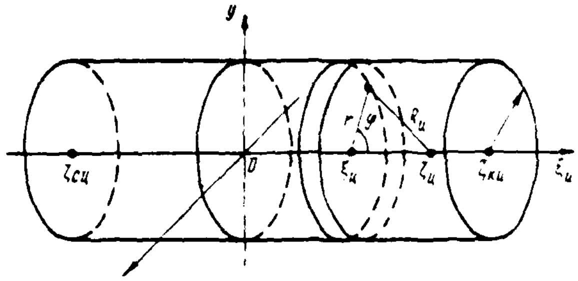

Fig. 4.3.

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By the replacement of the variable/alternating of the integration

$$\xi = \frac{1}{\gamma}\xi_{u}$$

taking into account (4.21) we will obtain

$$U(\zeta) = \frac{\gamma^2}{2\varepsilon_0}\int_{\zeta_c}^{\zeta_K} \varrho(\xi)\left[\sqrt{\frac{1}{\gamma^2}R_c^2 + (\zeta-\xi)^2} - \sqrt{(\zeta-\xi)^2}\right]d\xi.$$

In the laboratory system the difference longitudinal coordinate of particle ζ and the phase of particle ψ are connected with the relationship/ratio

$$\zeta = -\frac{v_s}{\omega}\psi.$$

Introducing the new replacement of the variable/alternating of the integration

$$\xi = -\frac{v_s}{\omega}\alpha,$$

we will finally have

$$U(\psi)=\frac{1}{2\varepsilon_0}\left(\frac{2c\gamma}{\omega}\right)^2\int\limits_{\psi}^{\psi_c}\varrho(\alpha)\times$$

$$\times\left[\sqrt{\left(\frac{\omega R_c}{\gamma v_s}\right)^2-(\psi-\alpha)^2}-\sqrt{(\psi-\alpha)^2}\right]d\alpha. \qquad (4.22)$$

Let us pass on the left side of equality (4.22) of the potential of the proper field of cluster to introduced above dimensionless potential function $\Phi(\psi)$

$$U(\psi)=\frac{\gamma v_s c}{e}\left(\frac{\Omega_c}{\omega}\right)^2[\Phi(\psi)-\Phi_0(\psi)]$$

and let us replace $\varrho(\psi)$ with its expression (4.19). As a result we will obtain integral equation for the potential function, which corresponds to self-congruent field,

$$\Phi(\psi)=\Phi_0(\psi)-h_\psi\int\limits_{\psi_K}^{\psi_c}R(\psi,\alpha)\sqrt{\Phi_c-\Phi(\alpha)}\,d\alpha. \qquad (4.23)$$

Function $\Phi_0(\psi)$ is assigned; it corresponds to the beam of zero intensity and is determined by expression (4.13). Kernel of integral equation

$$R(\psi,\alpha)=\sqrt{\left(\frac{\omega R_c}{\gamma v_s}\right)^2+(\psi-\alpha)^2}-\sqrt{(\psi-\alpha)^2}. \qquad (4.24)$$

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Constant parameter h_ψ is introduced as follows

$$h_\psi=\frac{I}{2I_\psi}[\Phi_c-\Phi(\psi_0)]^{-1/2}, \qquad (4.25)$$

the characteristic strength of current I_ψ depending onlv on the parameters of the accelerator

$$I_\psi=\pi^2\left(\frac{\Omega_c}{\omega}\right)^2\left(\frac{R_c}{\lambda}\right)^2\beta I_0. \qquad (4.26)$$

At the solution of integral equation (4.23) parameter h_ψ is conveniently been assigned. For each value/significance h_ψ from equation (4.23) and supplementary conditions (4.11), (4.12) are determined function $\Phi(\psi)$ and values ψ_c, ψ_s, Φ_c, also depending on h_ψ. After this it is possible to find the ratio of the peak current of cluster I to the characteristic value of current I_ψ, corresponding to given h_ψ. The method of the numerical solution of integral equation (4.23) with boundary conditions (4.11), (4.12) is described in work [127].

As it will be shown, parameter h_ψ is approximately proportional to phase current density on plane ψ, p_ψ, i.e., it is proportional to the ratio of complete peak beam current to the area, included by separatrix. Therefore by analogy with the Coulomb parameter of beam, introduced in Chapter 3, value h_ψ can be named the longitudinal Coulomb parameter of beam.

§ 4.2. Stability of the longitudinal vibrations of particles in the intense beams.

Let us first of all examine small longitudinal vibrations in the intense beams. Let us introduce potential function $\Phi(\psi)$ into the equations of motion (4.6), we will obtain

$$\frac{d\psi}{dt}=\frac{\omega}{\gamma^2 p_s v_s}p_\psi;$$

$$\frac{dp_\psi}{dt}=-\frac{\omega p_s v_s}{\gamma}\left(\frac{\Omega_c}{\omega}\right)^2\frac{d\Phi}{d\psi}(\psi).$$

Hence it is possible to obtain second order equation for the phase of particle ψ. In the conservative approximation/approach we have

$$\frac{d^2\psi}{dt^2}+\Omega^2\frac{d\Phi}{d\psi}(\psi)=0, \qquad (4.27)$$

where

$$\Omega=\gamma^{-3/2}\Omega_c$$

there is relativistic frequency of small longitudinal vibrations in the beam of zero intensity. Equation (4.27) is nonlinear. Assuming that the phase of particle little differs from synchronous phase, let us expand function $\frac{d\Phi}{d\psi}$ in series/row about point $\psi=\psi_0$ and we will be restricted to the linear terms of the expansion

$$\frac{d\Phi}{d\psi}(\psi)=\frac{d\Phi}{d\psi}(\psi_0)+\frac{d^2\Phi}{d\psi^2}(\psi_0)(\psi-\psi_0)+\dots$$

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According to expression (4.10), the constant of series/row is equal to zero. The linearized equation takes the form

$$\frac{d^2\psi}{dt^2}+\Omega^2\frac{d^2\Phi}{d\psi^2}(\psi_0)(\psi-\psi_0)=0. \qquad (4.28)$$

From equation (4.28) it follows that the oscillations of phase occur around the new state of equilibrium $\psi=\psi_0$ with a frequency of the small of

$$\Omega_{\text{кул}}=\Omega\sqrt{\frac{d^2\Phi}{d\psi^2}(\psi_0)}. \qquad (4.29)$$

As it is easy to be convinced, when $h_\psi=0$ we have

$$\frac{d^2\Phi_0}{d\psi^2}(0)=1.$$

and $\Omega_{кул} = \Omega$. The second derivative of potential function can be determined by dual differentiation of equality (4.23). Kernel of integral equation $R(\psi, \alpha)$ on diagonal $\psi = \alpha$ has a derivative discontinuity. Actually/really, from expression (4.24) follows

$$\frac{\partial R}{\partial \psi} = \frac{\psi - \alpha}{\sqrt{\left(\frac{\omega R_c}{\gamma v_s}\right)^2 + (\psi - \alpha)^2}} - \frac{\psi - \alpha}{|\psi - \alpha|};$$

thus,

$$\frac{\partial R}{\partial \psi} = \begin{cases} \frac{\psi - \alpha}{\sqrt{\left(\frac{\omega R_c}{\gamma v_s}\right)^2 + (\psi - \alpha)^2}} - 1 & \text{при } \alpha < \psi; \\ \frac{\psi - \alpha}{\sqrt{\left(\frac{\omega R_c}{\gamma v_s}\right)^2 + (\psi - \alpha)^2}} + 1 & \text{при } \alpha > \psi. \end{cases}$$

Key: (1). with.

Plotted function $R(\psi, \alpha)$ is given in Fig. 4.4.

From Fig. 4.4 it is evident that the kernel of integral equation has pulse character. This is explained by the fact that the Coulomb forces rapidly decrease with the distance, so that a basic effect on the given particle have only the adjacent particles. With the decrease of parameter $\frac{\omega R_c}{\gamma v_s}$ the peak of nucleus decreases on the height/altitude and it becomes narrower. In the limit, when $\frac{\omega R_c}{\gamma v_s} \to 0$, the nucleus vanishes evenly to relatively α.

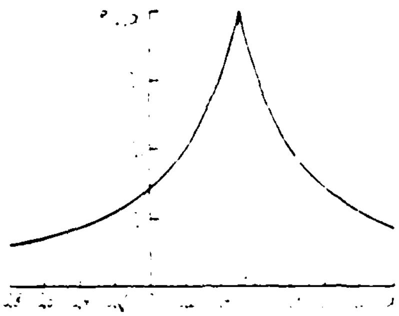

Fig. 4.4.

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During the differentiation of integral (4.23) with respect to the parameter the derivatives must be calculated separately in sections $\psi_\kappa\psi$ and $\psi\psi_c$. The second derivative of the nucleus

$$\frac{\partial^2 R}{\partial \psi^2} = \left(\frac{\omega R_c}{\gamma v_s}\right)^2 \left[\left(\frac{\omega R_c}{\gamma v_s}\right)^2 - (\psi - \alpha)^2\right]^{-3/2} \qquad (4.30)$$

gaps does not have. Differentiating twice equality (4.23) and substituting $\psi = \psi_0$, we obtain

$$\frac{d^2\Phi}{d\psi^2}(\psi_0) = \frac{d^2\Phi_0}{d\psi^2}(\psi_0) - 2h_\psi \sqrt{\Phi_c - \Phi(\psi_0)}\,(1 - M). \qquad (4.31)$$

Value M is determined by the integral

$$M = \frac{1}{2}\int_{\psi_\kappa}^{\psi_c} \frac{\sqrt{\Phi_c - \Phi(\alpha)}}{\sqrt{\Phi_c - \Phi(\psi_0)}} \frac{\partial^2 R}{\partial \psi^2}(\psi_0, \alpha)\, d\alpha. \qquad (4.32)$$

Expression (4.31) can be simplified, if to use the determination of longitudinal Coulomb parameter (4.25)

$$\frac{d^2\Phi}{d\psi^2}(\psi_0) = \frac{d^2\Phi_0}{d\psi^2}(\psi_0) - \frac{I}{I_\psi}(1 - M). \qquad (4.33)$$

by formulas (4.29), (4.33) is determined the dependence of the frequency of small longitudinal vibrations on the peak current of cluster.

For the numerical solution of integral equation (4.23) it is necessary to previously assign two values: by synchronous phase in the beam of zero intensity φ_s, of that being determining potential function $\Phi_0(\psi)$, and by parameter $\frac{\omega R_c}{\gamma v_s}$, by that being determining kernel of integral equation. As it was established in S 3.7, the ratio of the aperture of the strong-focusing accelerator to value $\beta\lambda$ during the optimum identification of the parameters of the focusing system virtually is uniquely determined by the assigned frequency of accelerating field, by energy of injection and by maximum permissible induction in the poles of magnetic quadrupole lenses. With λ=2 m and W_s=0.7 MeV the maximum amplitude of free transverse vibrations can be accepted equal to 0.9 cm. Assuming/setting the mean radius of beam by equal to $R_c = 0{,}5$ cm, for the initial part of the accelerator we will obtain $\frac{2\pi}{\gamma}\cdot\frac{R_c}{\beta\lambda} = 0.4$. Let us accept, further, $\cos\varphi_s = 0{,}8$. Are given below the basic results of the numerical solution of integral equation (4.23) at these values of the parameters.

Fig. 4.5 gives the dependence of the frequency of small longitudinal vibrations on Coulomb parameter h_φ. Frequency is calculated according to formulas (4.31), (4.29) after the determination of potential function $\Phi(\psi)$ for the series/row of values h_ψ. Frequency of small monotonically decreases with an increase in the Coulomb parameter and within the limit it vanishes when $h_\psi \to \infty$.

The behavior of potential function $\Phi(\psi)$ at different values is shown in Fig. 4.6.

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Level Φ_c with increase h_ψ increases, but this does not have vital importance, since the course of separatrix does not depend on absolute level function $\Phi(\psi)$. In the absence of Coulomb pushing apart the minimum of potential function, which corresponds to synchronous phase, lies/rests in the beginning of the coordinates: $\psi = 0$. With an increase in the Coulomb parameter the minimum weakly is displaced to the side of negative values ψ, depth and width of potential well decrease. In this case the decrease of potential well depth occurs considerably more rapid than the decrease of its width, and in the limit, when $h_\psi \to \infty$, the depth of pit vanishes, and width, as will be evident below, it remains final. At the infinite value/significance of Coulomb parameter h_ψ potential function must have straight/direct horizontal section and a singular point of the type of center on the plane will disappear.

With a change in the form of potential function is changed the course of separatrix. For the characteristic of the behavior of stability region at the high values of the Coulomb parameter it is expedient to introduce any auxiliary parameter, which remains final when $h_q \rightarrow \infty$ From formula (4.29) it follows that the process of phase stability is retained, until the values of the second derivative of potential function at the point of the minimum lie/rest in the limits

$$1 > \frac{d^2\Phi}{d\psi^2}(\psi_0) > 0.$$

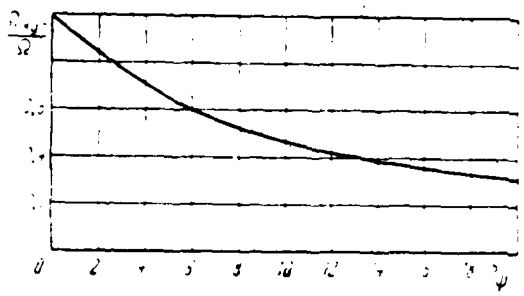

Fig. 4.5.

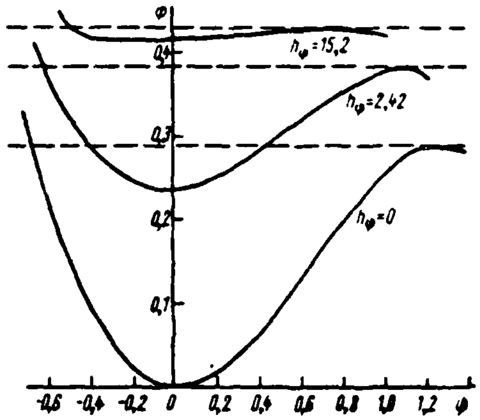

Fig. 4.6.

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Space-charge effect of beam on the process of phase stability is convenient to characterize with value

$$k = 1 - \frac{d^2\Phi}{d\psi^2}(\psi_0),$$

of that of unambiguously connected with the Coulomb parameter. In the absence of longitudinal pushing apart we have k=0, and the infinite value/significance of the Coulomb parameter corresponds then k=1. For the frequency of small we will obtain the expression

$$\Omega_{кул} = \Omega \sqrt{1-k}.$$

The dependence of the auxiliary parameter k on the longitudinal Coulomb parameter of beam h_ψ is given in Fig. 4.7.

The function

$$\mathcal{F}(\psi) = \sqrt{2\,[\Phi_c - \Phi(\psi)]} \qquad (4.34)$$

does not depend on relation $\frac{\Omega_c}{\omega}$ and, therefore, is determined separatrix for any frequencies of small phase oscillations, which occur when $h_\psi = 0$. On plane ψ $\frac{p-p_s}{p_s}$ the separatrix, according to expressions (1.50), (4.9), will take the form

$$g_c(\psi) = \gamma^{1/2}\,\frac{\Omega_c}{\omega}\,\mathcal{F}(\psi),$$

where g_c - relative difference in impulses/momenta/pulses (1.58). Fig. 4.8 gives plotted functions $\mathcal{F}(\psi)$ at the different values of parameter k, while in Fig. 4.9 - the dependence of stability limits ψ_c, $\psi_к$, $\mathcal{F}_{макс}$, of the abscissa of center ψ_0, and also relation $\Omega_{кул}/\Omega$ on k. These graphs/curves make it possible to trace how is deformed the stability region of longitudinal vibrations with an increase in the Coulomb parameter.

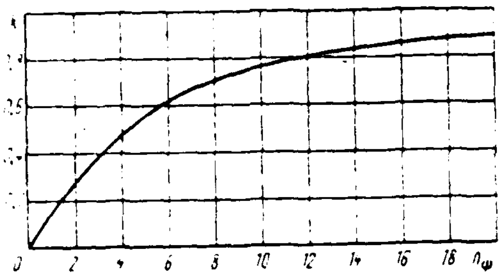

Fig. 4.7.

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With increase of k the stability region is reduced. Decrease occurs in essence due to the decrease of the permissible variations of particles on the impulses/momenta/pulses, moreover with k→1 ($h_\phi \to \infty$) the permissible variations on the impulses/momenta/pulses vanishes. The width of stability region with respect to the phase is decreased relatively slowly. Thus, with an increase in the auxiliary parameter k from zero from 0.86 (which corresponds to an increase in the Coulomb parameter from zero to $h_\phi = 15{,}2$) the vertical spread/scope of separatrix decreases five times, into the capture region on the phases it is compressed approximately/exemplarily to 30o/o. From the graphs/curves in Fig. 4.9 it follows that the stability region with respect to the phases remains final with k=1.

Current in the cluster is distributed along the longitudinal axis according to the law

$$I(\psi) = \pi R_c^2 v_s \varrho(\psi). \tag{4.35}$$

Clusters follow each other with the period 2π. Average/mean beam current is equal to

$$I_{cp} = \frac{1}{2\pi} \int_{\psi_K}^{\psi_c} I(\psi)\, d\psi. \tag{4.36}$$

After substituting into integral (4.36) of expression (4.35), (4.19), we will obtain the following formula for the ratio of average/mean beam current to the peak current of the cluster

$$\frac{I_{cp}}{I} = \frac{1}{2\pi} \int_{\psi_K}^{\psi_c} \frac{\sqrt{\Phi_c - \Phi(\psi)}}{\sqrt{\Phi_c - \Phi(\psi_0)}}\, d\psi. \tag{4.37}$$

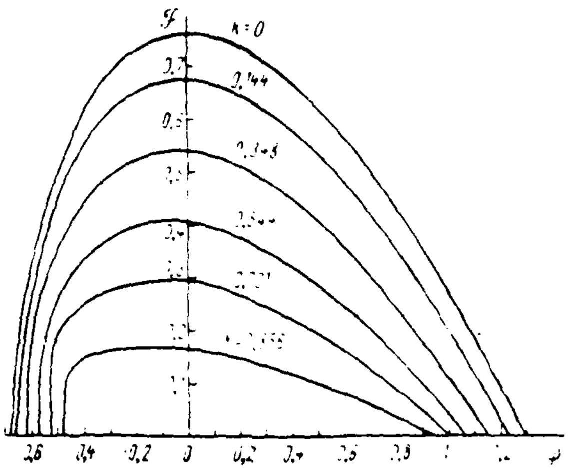

Fig. 4.8.

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For calculating interaction of beam with accelerating field it is necessary to know the amplitudes of the fundamental harmonics of the current of cluster in the form of Fourier series

$$I(\psi) = I_{cp} + I_c \cos\psi + I_s \sin\psi + \ldots$$

Coefficients of the fundamental harmonics of the series/row

$$I_c = \frac{1}{\pi}\int_{\psi_{\text{н}}}^{\psi_c} I(\psi)\cos\psi\, d\psi;$$

$$I_s = \frac{1}{\pi}\int_{\psi_{\text{н}}}^{\psi_c} I(\psi)\sin\psi\, d\psi.$$

Utilizing expressions (4.35), (4.19), we will obtain the ratio of the amplitude of the first cosinusoidal harmonic to the peak current

$$\frac{I_c}{I} = \frac{1}{\pi}\int_{\psi_{\text{н}}}^{\psi_c} \frac{\sqrt{\Phi_c - \Phi(\psi)}}{\sqrt{\Phi_c - \Phi(\psi_0)}}\cos\psi\, d\psi.$$

For the first sinusoidal harmonic of Fourier series we have

$$\frac{I_s}{I} = \frac{1}{\pi}\int_{\psi_{\text{н}}}^{\psi_c} \frac{\sqrt{\Phi_c - \Phi(\psi)}}{\sqrt{\Phi_c - \Phi(\psi_0)}}\sin\psi\, d\psi.$$

In view of the fact that the extent of cluster along the axis of phases $\psi_c - \psi_H$ little is changed with an increase in the Coulomb parameter, one should expect that the ratio of average/mean beam current to the peak weakly depends on h_ψ. The considerations indicated occur also for the ratio of the amplitudes of the fundamental harmonics to the peak beam current. Precise numerical calculations confirm this fact. Fig. 4.10 gives the graphs/curves, which characterize the dependence of ratios $\frac{I_{cp}}{I}$, $\frac{I_c}{I}$ and $\frac{I_s}{I}$ on the auxiliary parameter k. From Fig. 4.10 it is evident that the coefficients of Fourier-expansion of beam current approximately linearly depend on parameter k, moreover dependence itself is weak. Good approximations/approaches in entire range of change k in parameters $\varphi_s, \frac{\omega R_c}{\gamma v_c}$ placed into the calculation give the following approximations:

$$\frac{I_{cp}}{I} = 0{,}215\,(1 - 0.25k);$$
$$\frac{I_c}{I} = 0{,}381\,(1 - 0.19k); \qquad (4.38)$$
$$\frac{I_s}{I} = 0{,}070\,(1 - 0{,}92k).$$

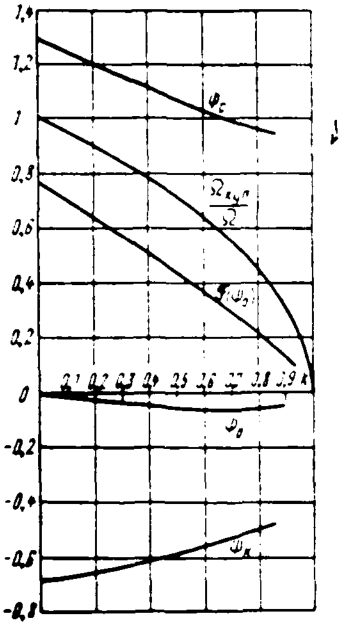

Fig. 4.9.

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Thus, the value of the integral

$$\Lambda = \int_{\psi_{\kappa}}^{\psi_c} \frac{\sqrt{\Phi_c - \Phi(\psi)}}{\sqrt{\Phi_c - \Phi(\psi_0)}}\, d\psi. \qquad (4.39)$$

connected with the average/mean beam current with the equality

$$I_{cp} = I \frac{\Lambda}{2\pi}. \qquad (4.40)$$

noncritically depends on k. With k→1 function $\frac{\Phi(\psi)}{\Phi_c}$ in appropriate interval $\psi_{\kappa}\psi_c$ evenly approaches unity, so that value Λ approaches the limiting value of difference $\psi_c - \psi_{\kappa}$

Let us examine connection/communication of peak current with the longitudinal Coulomb parameter of beam. Fig. 4.11 shows the dependence of peak current and average/mean beam current on parameter k. At the low values of parameter k it is as follows connected with the peak current

$$k = 0.18 \frac{I}{I_{\psi}}. \qquad (4.41)$$

With an increase k this dependence differs from linear law. When $k \to 1 \ (h_{\psi} \to \infty)$ peak current approaches finite value. Actually/really, according to expression (4.33),

$$\frac{I}{I_{\psi}} = \frac{\Phi_0^{*}(\psi_0) - \Phi^{*}(\psi_0)}{1 - M}. \qquad (4.42)$$

The numerator of the right side of this equality when $h_{\psi} \to \infty$ remains the value of final.

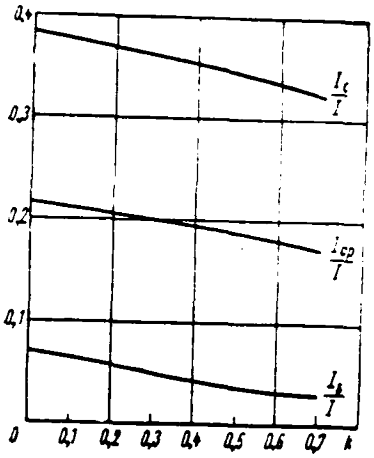

Fig. 4.10.

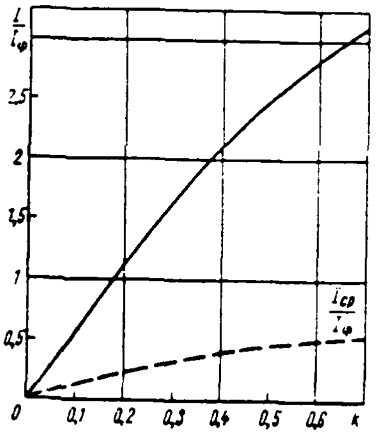

Fig. 4.11.

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On the other hand, from formula (4.32) it follows that when $h_c \to \infty$

$$\lim M = M_\infty = \frac{1}{2}\int_{\psi_к}^{\psi_c} \frac{\partial^2 R}{\partial \psi^2}(\psi_0, \alpha)\, d\alpha.$$

After substituting function (4.30) and after producing integration, we will obtain

$$M_\infty = \frac{1}{2}\left[\frac{\psi_c-\psi_0}{\sqrt{\left(\frac{\omega R_c}{\gamma v_s}\right)^2+(\psi_c-\psi_0)^2}}+\frac{\psi_к-\psi_0}{\sqrt{\left(\frac{\omega R_c}{\gamma v_s}\right)^2+(\psi_к-\psi_0)^2}}\right]. \quad (4.43)$$

The limiting value M<1, so that the denominator of expression (4.42) in the limit does not become zero. With k=1 we have

$$I_{пред} = \frac{\Phi_0''(\psi_0)}{1-M_\infty} I_\psi.$$

Since ψ_0 remains close to zero (see Fig. 4.9), $\Phi_0^{*}(\psi_0)\approx 1$ and

$$I_{\text{пред}} \approx \frac{I_\psi}{1-M_\infty}. \qquad (4.44)$$

Extrapolation of computed values of M to k=1 gives following limiting value $M_\infty = 0.74$. Hence it follows that maximum peak current in cluster $I_{\text{пред}} = 3.85\, I_\psi$.

Let us define the longitudinal phase volume of beam V_ψ as the area, occupied with the representative points of particles on the plane of canonical-conjugated/combined variable/alternating ψ, $p_\psi/\mathscr{E}_0$:

$$V_\psi = \frac{1}{\mathscr{E}_0}\int\int d\psi\, dp_\psi.$$

In the case in question entire/all region, included by separatrix, is filled with particles. The area of this region, according to expression (4.9), is equal to

$$V_\psi = 2\sqrt{2}\beta^2\gamma^{3/2}\frac{\Omega_c}{\omega}\int\limits_{\psi_n}^{\psi_c}\sqrt{\Phi_c-\Phi(\psi)}\,d\psi.$$

Let us multiply and let us divide the right side of equality (4.25) on V_ψ. Taking into account formula (4.39) we will obtain the following expression for the longitudinal Coulomb parameter of the beam:

$$h_\psi = \sqrt{2}\beta^2\gamma^{3/2}\frac{\Lambda\Omega_c}{\omega I_\psi}\cdot\frac{I}{V_\psi}. \qquad (4.45)$$

The ratio of complete peak beam current to its longitudinal phase volume let us name/call phase current density on plane ψ, p_ψ ε_n. Since Λ little differs from constant value in entire range of change h_ψ from 0 to ∞, the Coulomb parameter of beam, introduced by equality (4.25), is approximately proportional to phase current density.

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Thus, the obtained dependence between the sizes/dimensions of capture region and Coulomb parameter h_ψ (or auxiliary parameter k) reflects the dependence of these sizes/dimensions on the phase current density of the injected beam. Passage to the limit $h_\psi \to \infty$ — this is transition to the infinitely high phase current density; in this case the beam current, seized into acceleration mode, remains final, and longitudinal phase volume vanishes.

Since the real beam, injected into the linear accelerator, possesses nonzero phase volume, the maximum peak current of the particles, seized into acceleration mode (4.44), can be in the principle realized only with the infinite current of injection. picture appears as follows. With an increase in the current of injection the peak current of the seized particles monotonically increases. With the low currents of injection the peak current of the seized particles increases approximately proportional to the injected current. Subsequently an increase in the peak current of the seized particles begins to lag behind an increase in the current of injection and approaches the finite value with an infinite increase in the injected current. In the extreme case into acceleration mode are seized only the particles whose energy coincides with the energy of synchronous particle. The average/mean current of the accelerated beam also monotonically increases with an increase in the phase current density of injection and within the limit it approaches the finite quantity

$$\lim I_{cp} = \frac{\Lambda}{2\pi(1-M_{\infty})} I_{\Phi}.$$

At values φ_s, $\frac{\omega R_c}{\gamma v_s}$, of those used in the numerical calculation, limiting mean beam current, determined by longitudinal pushing apart, it is possible to obtain from first equality (4.38); it is equal to $I_{cp} = 0{,}6 I_{\Phi}$.

In short-wave ionic strong-focusings accelrator the current of the accelerated beam is limited not to longitudinal, but transverse pushing apart. Let us examine proton accelerator with the standard parameters: λ=2m; β_s=0.04; $\cos\varphi_s = 0{,}8$; $W_\lambda = 2{,}7 \cdot 10^{-3}$. For this accelerator $I_{\Phi} = 650$ ma, whence follows $I_{\text{пред}} = 2{,}5$ a. The maximum value/significance of the peak current, determined by transverse pushing apart, for the accelerator with the same parameters was obtained in § 3.7 and proved to be equal to $I_{\text{max}} \approx 520$ mA. With the decrease of peak current in the cluster the longitudinal Coulomb parameter of beam rapidly falls. If we assume I=600 mA, then from formula (4.41) follows k=0.16 (or $h_\varphi = 0.9$). The capture region of particles into acceleration mode is decreased in this case on the phase only by 5o/o, and on the relative scatter of impulses/momenta/pulses on 14o/o; the frequency of small longitudinal vibrations falls on 8o/o. One should recall that the obtained numerical ratios relate to the idealized case when all particles, not seized into acceleration mode, are considered fallen of the beam. The account of such particles can substantially raise the theoretical value/significance of the limiting current, determined by longitudinal pushing apart.

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§ 4.3. Dependence of the limiting current, determined by longitudinal pushing apart, on the parameters of accelerator.

The solutions of integral equation (4.23) depend on three parameters: h_φ, φ_s, $\frac{\omega R_c}{\gamma v_s}$. In § 4.2 are given the results of a precise numerical solution of equation (4.23) for the case when the values of two parameters φ_s and $\frac{\omega R_c}{\gamma v_s}$ are fixed/recorded. These solutions made it possible to explain the general character of the strains of potential function and separatrix with an increase in the phase current density of beam on plane ψ, p_ψ. However, all quantitative results related only to the selected particular values of parameters φ_s, $\frac{\omega R_c}{v_s}$ and it did not make it possible to explain how are changed solutions during the variation of the values indicated. The relationships/ratios, which describe space-charge effect on the longitudinal vibrations of particles, prove to be simpler and easily more forseeable, if one assumes that synchronous phase φ_s and the phase of particle ψ they are small in the absolute value

$$|\varphi_s| \ll 1; \quad |\psi| \ll 1.$$

In this case it is possible to lower a number of independent parameters in equation (4.23) up to two, which substantially simplifies task. The requirement of the smallness of values ψ and φ_s is not strong. The power series obtained with the expansion of trigonometric functions rapidly converge, and already the second approximation/approach proves to be sufficiently to precise ones. Virtually it suffices to require satisfaction of conditions $|\varphi_s| < 1$; $|\psi| < 1$.

Having expanded function (4.13) in the power series of ψ and φ_s and after being restricted by the members of the third degree, we will obtain

$$\Phi_0(\psi) = -1 - \frac{1}{2}\psi^2 + \frac{\psi^3}{6\varphi_s}.$$

Let us introduce the new variable/alternating

$$x=\frac{\psi}{|\varphi_s|};\quad y=\frac{\alpha}{|\varphi_s|}.$$

Then

$$\Phi_0(\psi)=\varphi_s^2\Phi_0^*(x)-1, \qquad (4.46)$$

where

$$\Phi_0^*(x)=\frac{1}{2}\left(x^2-\frac{1}{3}x^3\right). \qquad (4.47)$$

Coordinates of center, saddle and second boundary of the region of phase stability in the new variable/alternating

$$x_0=\frac{\psi_0}{\varphi_s};\quad x_c=\frac{\psi_c}{\varphi_s};\quad x_\kappa=\frac{\psi_\kappa}{|\varphi_s|}.$$

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By analogy with expression (4.46) let us introduce into the examination new potential function $\Phi^*(x)$, connected with function $\Phi(\psi)$ with the equality

$$\Phi(\psi)=\varphi_s^2\Phi^*(x)-1. \qquad (4.48)$$

Let us accept, furthermore, the following designations:

$$\Phi_c^* = \Phi^*(x_c); \qquad (4.49)$$

$$\Lambda^* = \int_{x_0}^{x_c} \sqrt{\frac{\Phi_c^* - \Phi^*(x)}{\Phi_c^* - \Phi^*(x_0)}}\, dx. \qquad (4.50)$$

After passing in integral (4.39) to the new variable/alternating, we will obtain

$$\Lambda^* = \frac{\Lambda}{\varphi_s}.$$

Yard integral equation (4.24) in the new variable/alternating it is reduced to the form

$$R(\psi, \alpha) = \varphi_s \left[\sqrt{\Gamma^2 - (x-y)^2} - \sqrt{(x-y)^2}\right]. \qquad (4.51)$$

Here

$$\Gamma = \frac{1}{\gamma} \cdot \frac{R_c}{\frac{\varphi_s}{2\pi} - \beta\lambda}. \qquad (4.52)$$

Parameter Γ makes simple geometric sense. This parameter is proportional to the ratio of a transverse radius of cluster to the longitudinal length of cluster in the beam of zero intensity. It is obvious that with the decrease of value Γ the forces of longitudinal Coulomb pushing apart must decrease, since cluster in this case is extracted lengthwise.

Let us pass in integral equation (4.23) to the new variable/alternating of integration y. Taking into account formulas (1.62), (4.46), (4.48)-(4.51) let us arrive at the equation for potential function Φ^*

$$\Phi^*(x)=\Phi_0^*(x)-h_\varphi^*\int_{x_H}^{x_c} R^*(x,y)\sqrt{\Phi_c^*-\Phi^*(y)}\,dy. \qquad (4.53)$$

Kernel of equation (4.53) has the form

$$R^*(x,y)=\frac{2}{\Gamma^2}\left[\sqrt{\Gamma^2+(x-y)^2}-\sqrt{(x-y)^2}\right]. \qquad (4.54)$$

Coulomb parameter of equation (4.53)

$$h_\varphi^*=\frac{1}{2}\Gamma^2|\varphi_s|h_\varphi \qquad (4.55)$$

or

$$h_\varphi^*=\frac{4\sqrt{\pi}\Lambda^*}{I_0\sqrt{\beta\gamma W_\lambda|\mathrm{tg}\,\varphi_s|}}\cdot\frac{I}{V_\varphi}. \qquad (4.56)$$

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In the approximation/approach accepted the equation of separatrix is converted as follows

$$\mathcal{F}(x)=|\varphi_s|\sqrt{2[\Phi_c^*-\Phi^*(x)]}.$$

For the beam of zero intensity $x_c=2;\ \Phi_c^*=2/3$. Hence

$$\mathcal{F}_0(x)=\frac{|\varphi_s|}{\sqrt{3}}(2-x)\sqrt{x+1}. \qquad (4.57)$$

Function (4.57) is given in Fig. 4.12. In the same figure for the comparison dotted line showed the same function, calculated according to strict formula (1.59a)

$$F_0(\psi)=\sqrt{2}\sqrt{1+\cos\psi-(\psi-\sin\psi+2\varphi_s)\,\mathrm{ctg}\,\varphi_s}.$$

Both functions are calculated when $\varphi_s=-0.65$. From Fig. 4.12 it is evident that approximation (4.47) transfers well the behavior of potential function.

The giving rise to function of integral equation (4.53) does not depend on synchronous phase. Hence it follows that the solutions of integral equation (4.53) at the given values of parameters Γ, h_φ^* remain valid during any selection of synchronous phase. This considerably facilitates the numerical integration of equation and simplifies the analysis of solutions. It is obvious that also value Λ^* on the selection of synchronous phase does not depend. Therefore the formula, which connects peak tone with the average/mean beam current,

$$I_{cp}=\frac{|\varphi_s|}{2\pi}\Lambda^* I \qquad (4.40a)$$

in the approximation/approach accepted is accurate at any values φ_s. For the beam of the zero intensity

$$\Lambda^*=\frac{1}{2}\int_{-1}^{2}(2-x)\sqrt{x+1}\,dx,$$

whence $\Lambda^*=\frac{6\sqrt{3}}{5}$.

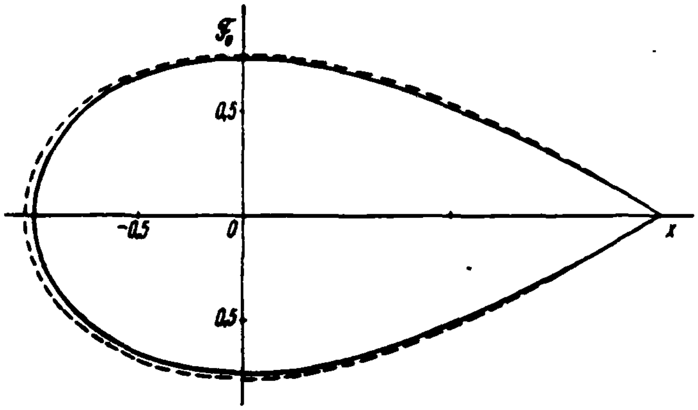

Fig. 4.12.

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Thus, for the beam of the noninteracting particles

$$I_{cp} = 1{,}04 \frac{|\varphi_s|}{\pi} I. \qquad (4.58)$$

Analogous with expression (4.31) as a result of dual differentiation of equality (4.53) we will obtain

$$\frac{d^2\Phi_0^*}{dx^2}(x_0) - \frac{d^2\Phi^*}{dx^2}(x_0) = \frac{4h_\psi^*}{\Gamma^2} \sqrt{\Phi_c^* - \Phi^*(x_0)}\,(1 - M^*), \qquad (4.59)$$

where

$$M^* = \frac{\Gamma^2}{4} \int_{x_\kappa}^{x_c} \frac{\sqrt{\Phi_c^* - \Phi^*(y)}}{\sqrt{\Phi_c^* - \Phi^*(x_0)}} \cdot \frac{\partial^2 R^*}{\partial x^2}(x_0, y)\,dy.$$

By the replacement of the variable/alternating of integration

$\alpha = \eta \varphi_s$ it is easy to show

$$M^*(x_0, x_c, x_K, \Gamma) \equiv M\left(\psi_0, \psi_c, \psi_K, \varphi_s, \frac{\omega R_c}{V \varphi_s}\right).$$

It is evident that at the given value of shape factor Γ value M on the selection of synchronous phase does not depend. From expression (4.43) and latter/last identity it follows

$$M_\infty = \frac{1}{2}\left[\frac{x_c - x_0}{\sqrt{\Gamma^2 - (x_c - x_0)^2}} + \frac{x_K - x_0}{\sqrt{\Gamma^2 - (x_K - x_0)^2}}\right]. \quad (4.60)$$

Since from relationships/ratios (4.25), (4.55), (4.59) we have

$$I = \left[\frac{d^2\Phi_0^*}{dx^2}(x_0) - \frac{d^2\Phi^*}{dx^2}(x_0)\right]\frac{I_\varphi}{1 - M^*},$$

that hence it follows that formula (4.44) determines maximum peak beam current at given values Γ, I_φ independent of the value of synchronous phase. Obtained above numerical relationship/ratio $I_{пред} = 3.85\ I_\varphi$ is correct for $\Gamma = 0.515$. With the decrease of shape factor Γ value M_∞ monotonically grows/rises. In this case M_∞ approaches unity and limiting current with given one I_φ increases.

The characteristic value of current I_φ (4.26) can be represented in the form

$$I_\varphi = \frac{\pi}{2} W_\lambda \ \mathrm{tg}\,\varphi_s \left(\frac{R_c}{\lambda}\right)^2 I_0, \tag{4.61}$$

and it on energy of particles does not depend. However, with an increase in the energy of particles increases the longitudinal length of clusters and falls shape factor Γ (4.52), which raises the limiting value of peak current. Thus, Coulomb pushing apart has the greatest effect on longitudinal vibrations as to the transverse ones, with the low energies. The estimations, connected with space-charge effect on the process of phase stability, should be carried out for the energy of injection.

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An increase in the energy of injection makes it possible to raise the limiting current of beam and, correspondingly, to lower the effect of Coulomb interaction on the value of the capture region of particles into acceleration mode.

The limiting current of beam grows/rises also with an increase in the specific acceleration. As can easily be seen, this is connected with the fact that an increase in the specific acceleration with the assigned energy of injection leads to an increase in the frequency of longitudinal vibrations.

The effect of longitudinal Coulomb pushing apart very susceptibly/critically depends on selection of synchronous phase. With the decrease of the absolute value of synchronous phase is decreased the longitudinal length of clusters, which leads to an increase in shape factor Γ. On the other hand, decrease φ_s causes a drop in the frequency of longitudinal vibrations and corresponding the decrease of the characteristic value of current. Both these of effect act in one direction and lead to the considerable decrease of limiting current. At the low absolute values of synchronous phase longitudinal Coulomb pushing apart can prove to be the basic reason for the limitation of beam current in the accelerator.

The mean radius of beam R_c is proportional to the aperture of accelerator a. As shown in § 3.7, with an increase in the wavelength of the accelerating quotient field a/λ can be within some limits increased proportionally λ. Therefore the characteristic value of current (3.61), other conditions being equal, will increase proportional to λ^2. However, from expression (4.52) it is evident that shape factor Γ will in this case with an increase in the wavelength of accelerating field grow/rise, which impedes an increase in limiting current (4.44). For the evaluation of combined effect of both oppositely acting effects we convert expression (4.60), after assuming $\Gamma^2 \ll 1$. After expanding right side (4.60) into series about to degrees Γ^2 and after being restricted by member, linear relatively Γ^2, we will obtain

$$M_\infty = 1 - \frac{\Gamma^2}{4}\left[\frac{1}{(x_c - x_0)^2} + \frac{1}{(x_\kappa - x_0)^2}\right]. \qquad (4.62)$$

In the same time from equality (4.61) it follows

$$I_\varphi = \frac{\Gamma^2}{8\pi} W_\lambda \varphi_s^2 \,|\operatorname{tg} \varphi_s|\, \beta^2 \gamma^2 I_0. \qquad (4.63)$$

Thus, relation $I_{\varphi}(1-M_{\infty})$ on the shape factor in the first approximation, does not depend and, therefore, change $R_c\lambda$ other conditions being equal, the limiting current does not affect.

Let us assume $\mathrm{tg}\,\varphi_s \approx \varphi_s,\ x_0 \approx 0$ and let us substitute expressions (4.62), (4.63) into equality (4.44):

$$I_{\text{пред}} \approx \frac{W_\lambda}{2\pi} |\varphi_s|^3 \beta^2\gamma^2 I_0 \frac{x_c^2 x_K^2}{x_c^2 - x_K^2}.$$

By the extrapolation of graphs/curves in Fig. 4.9 to k=1 for limiting values $x_c,\ x_K$ when $\Gamma = 0.615$ and $\varphi_s = -0.65$ we obtain $x_c = 1.43;\ x_K = -0.66$. Then

$$I_{\text{пред}} \approx \frac{W_\lambda}{5\pi} \varphi_s{}^3\beta^2\gamma^2 I_0. \tag{4.64}$$

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A change in the wavelength of accelerating field does not directly affect limiting current, determined by longitudinal pushing apart. Nevertheless with an increase λ appears the possibility to raise specific acceleration, which with the conservation of energy of injection makes it possible to raise the limiting value of current and, therefore, to decrease space-charge effect on the capture region. Expression (4.64) makes it possible to quantitatively rate/estimate the effect of the selection of synchronous phase on the maximum intensity of beam. Let us return for the numerical example, given at the end of § 4.2. If we decrease the synchronous phase from 37 to 20°, then with the retention/preservation/maintaining the given values of energy of injection and specific acceleration limiting current decreases approximately six times and longitudinal Coulomb pushing apart it proves to be the already basic factor, which limits beam current. Therefore any methods of focusing, which require the considerable decrease of the absolute value of synchronous phase, lead to a reduction in the maximally possible intensity of beam.

§ 4.4. Effect of beam on accelerating field.

The acceleration of the beams of high intensity advances number of radio engineering problems, since interaction of beam with the field leads not only to the effect of particle acceleration, but also to a change in the parameters of accelerating field. The accelerated beam is grouped into the clusters, which follow with the frequency of accelerating field. These clusters aim on the elements of the accelerating system supplementary high-frequency the current, which leads to a change in amplitude and phase of field. In the principle occurs also the detuning of resonators with the load by their beam. On the stability of amplitude and phase of accelerating voltage are superimposed the close tolerances. Therefore the effects, connected with the effect of intense beam on accelerating field, can lead to the limitations of accelerated current. At the selection of the diagrams of the supply of the accelerating devices/equipment and diagrams of the automatic control of the parameters of field one should consider the effects indicated.

We will be restricted to the case when the resonance elements of the accelerating system are supplied from the separately-excited generator. The given calculations can be attributed to any construction/design of the accelerating system. However, for the concreteness is examined the accelerating system with drift tubes.

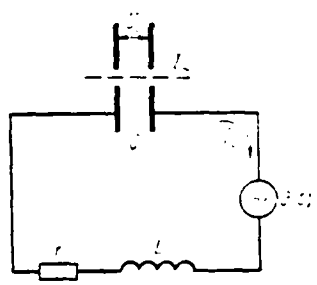

Fig. 4.13.

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Let us replace one it cut off cavity resonator by equivalent duct/contour with the concentrated elements/cells let us examine the capacitor/condenser of duct/contour, threaded by the bunched beam (Fig. 4.13). To account for the high-frequency power, introduced into the cavity resonator, with the duct/contour is connected equivalent source emf. It cut off cavity resonator it is characterized by three parameters: by natural frequency ω; by the quality

$$Q=\omega\frac{W_э}{P_м}, \qquad (4.65)$$

where $W_э$ — average/mean within the period stored up energy in the electric field, $P_м$ — high-frequency power of ohmic losses; shunt - by the impedance

$$Z=\frac{V^2}{2P_м}. \qquad (4.66)$$

where V - amplitude value of stress/voltage on the accelerating clearance. The parameters of equivalent duct/contour are connected with these values with the relationships/ratios

$$C=\frac{Q}{\omega Z};\quad L=\frac{Z}{\omega Q};\quad r=\frac{Z}{Q^2}. \qquad (4.67)$$

Let q(t) - the charge, induced to the plates of capacitor/condenser during this instantaneous location of the charges of beam. Regarding, value

$$I_{нав}(t)=\frac{dq(t)}{dt}$$

is called the induced current. The full current through the capacitor/condenser is equal to the sum of the bias current and induced current:

$$I_L=C\frac{dU_c}{dt}+\frac{dq}{dt}.$$

From the theorem Shockley-Ramo [128, 129] escape/ensues following connection/communication between the induced current and the beam current, which penetrates parallel-plate capacitor [130],

$$I_{\text{нав}}(t)=\frac{1}{g}\int_0^g I_{\sim}(z,\ t)\,dz. \tag{4.68}$$

Here g - distance between the plates of capacitor/condenser (approximately equal to the length of the accelerating clearance): $I_{\sim}(z,t)$ — the alternating current component of beam in the function of longitudinal coordinate and time. According to expression (4.68), the induced current at the given moment/torque t is equal to the value/significance average/mean at the gap length of the variable part of the beam current at the same moment/torque t.

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Current in duct/contour I_L and voltage across capacitor U_c they satisfy the equations

$$\begin{aligned} L\frac{dI_L}{dt}+rI_L+U_c=\mathcal{E}(t);\\ I_L=C\frac{dU_c}{dt}+I_{\text{нав}}(t), \end{aligned} \tag{4.69}$$

where $\mathcal{E}(t)$ — electromotive force of equivalent generator (Fig. 4.13). the induced current in the general case - sum of two components. First component I°(t) does not depend on accelerating voltage U_c; the second depends on amplitude and phase of accelerating voltage. In the linear approximation/approach this dependence can be represented in the form

$$I_{\text{нав}}(t)=I^0(t)+\frac{1}{R_{\text{п}}}U_c+C_{\text{п}}\frac{dU_c}{dt}. \tag{4.70}$$

Coefficients $\frac{1}{R_п}$, $C_п$ - some dimensional values whose values will be calculated below. After the substitution of expressions (4.67), (4.70) in equations (4.69), we will obtain

$$\frac{d^2U_c}{dt^2}+\frac{\omega}{Q_{экв}}\cdot\frac{dU_c}{dt}+\omega_1^2U_c=\frac{\omega}{\omega C_пZ}\left[\omega Q\mathcal{E}(t)-Z\left(\frac{dI^0}{dt}+\frac{\omega}{Q}I^0\right)\right], \quad (4.71)$$

where

$$\omega_1^2=\omega^2\frac{1+\frac{Z}{Q^2R_п}}{1+\frac{\omega C_пZ}{Q}}; \quad (4.72)$$

$$\frac{1}{Q_{экв}}=\frac{1}{Q}+\frac{Z}{R_п(Q+\omega C_пZ)}. \quad (4.73)$$

Thus, if the induced current depends substantially on voltage across capacitor, then duct/contour is disturbed/detuned. In this case changes also the energy factor of duct/contour.

The induced current to calculate more simply, if we in expression (4.68) replace the variable/alternating of integration. Let us introduce instead of the longitudinal coordinate z new variable/alternating

$$\tau=t-\int_0^z\frac{dz}{v(z,t)}, \quad (4.74)$$

where v - the longitudinal velocity of particle in the clearance. Variable τ - is the moment/torque of the time when at the entrance of the accelerating clearance appeared the particle, which achieved coordinate z at a given moment in time t. In S 4.2 is obtained the distribution of prompt current along the cluster

$$I(\psi) = I_{cp} + I_{\sim}(\psi).$$

Subsequently all phases we will count off from the moment/torque when attains maximum voltage across capacitor

$$U_c^0 = V_0 \cos \omega t$$

in the duct/contour, not loaded with beam.

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Phase ψ can be represented in the form

$$\psi = \omega\tau - \varphi_1,$$

where φ_1 - phase of the unloaded field at the moment/torque when at the entrance of clearance appears synchronous particle. Hence it is apparent that current I depends only on variable/alternating τ. Since time t in integral (4.68) is fixed/recorded, dz=-vτ and integral (4.68) is reduced to following form [130]

$$I_{\text{нав}}(t) = \frac{1}{g} \int_{\tau_g}^{t} I_{\sim}(\tau)\, v(t, \tau)\, d\tau. \qquad (4.75)$$

Value τ_g relates to the particle, which reaches at moment t of the second edge of clearance z=g

$$\tau_g = t - \int_0^g \frac{dz}{v}.$$

Disregarding the scatter of longitudinal velocities, let us suppose that all particles fly into the clearance with one and the same average speed v_0, equal to the velocity of the motion of cluster as whole. Voltage across capacitor, loaded with beam,

$$U_c(t) = V\cos(\omega t - \Theta). \qquad (4.76)$$

In the unloaded duct/contour $V=V_0$; $\theta=0$. In the transient mode/conditions amplitude and the phase of stress/voltage (4.76) - the functions of time, slow in comparison with the period of high-frequency field. We will be restricted to nonrelativistic approximation/approach. Particle motion in the clearance satisfies the equation

$$\frac{d^2z}{dt^2} = \frac{eV}{m_0 g}\cos(\omega t - \Theta),$$

moreover $\frac{dz}{dt} = v_0$ with $t=\tau$. Hence

$$v(t, \tau) = v_0[1 + \varepsilon u(t, \tau)], \qquad (4.77)$$

where

$$\varepsilon = \frac{eV}{m_0 g \omega v_0};$$
$$u(t, \tau) = \sin(\omega t - \Theta) - \sin(\omega\tau - \Theta).$$

Dimensionless parameter ε can be written in the form

$$\varepsilon = \frac{V}{4\pi\alpha U_0}, \qquad (4.78)$$

where α - coefficient of clearance (1.23); U_0 - current energy of synchronous particle, expressed in the electron volts:

$$U_s = \frac{W_s}{e}.$$

In the linear accelerators, as a rule, $\varepsilon \ll 1$. Let us calculate the fundamental harmonic of the induced current, holding the members of the zero and first order of the smallness of relatively low parameter ε.

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From equalities (4.38) it is evident that at the constant phase density within the separatrix the amplitude of the odd harmonic of current is small in comparison with the average/mean beam current and with the amplitude of even harmonic. We will obtain accuracy completely sufficient for our purposes, if let us assume that the cluster is symmetrical relative to synchronous particle. We approximate the law of current distribution along the cluster by the following even function

$$I(\psi) = \begin{cases} 0 \ \overset{(1)}{\text{при}} \ \psi < -\varphi_s; \\ I \quad -\varphi_s < \psi < \varphi_s; \\ 0 \quad \psi > \varphi_s. \end{cases}$$

Key: (1). with.

where I - peak beam current. The coefficients of the first three members of Fourier series

$$I(\psi) = I_{cp} + \sum_{n=1}^{\infty} I_n \cos n\psi$$

in this case will take the form

$$I_{cp} = \frac{\varphi_s}{\pi} I; \quad I_1 = \frac{2\sin \varphi_s}{\pi} I; \quad I_2 = \frac{\sin 2\varphi_s}{\pi} I. \qquad (4.79)$$

When $\varphi_s = 0{,}65$ we will obtain the following values for the average/mean current and the amplitudes of the fundamental harmonics: $I_{cp} = 0{,}206I$; $I_c = I_1 = 0{,}385I$; $I_s = 0.$ which is close to values (4.38). The alternating current component of the beam

$$I_{\sim}(\tau) = I_1 \cos(\omega\tau - \varphi_1) + I_2 \cos 2(\omega\tau - \varphi_1) + \ldots \qquad (4.80)$$

Further, if moment/torque t is assigned, then the lower limit of integral (4.75) is determined in the first approximation, by the equation

$$\tau_g = t - \frac{g}{v_0} + \frac{\varepsilon}{v_0} \int_0^g u(z, \tau_g)\, dz. \qquad (4.81)$$

Transition from one argument t to the next z in function $u(t, \tau_0)$ can be completed, using zero approximation to equality (4.74)

$$t = \tau + \frac{z}{v_0},$$

since function u(z, τ) it enters into the terms, which are of the order ε. Thus,

$$u(z, \tau) = \sin\left(\omega\tau + \frac{\omega z}{v_0} - \Theta\right) - \sin(\omega\tau - \Theta).$$

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After substituting latter/last expression into equation (4.81) and after holding down/retaining terms not higher than the first order of smallness, we will obtain

$$\tau_g = \tau_g^0 + \varepsilon\Delta\tau_g, \tag{4.82}$$

where

$$\tau_g^0 = t - \frac{g}{v_0};$$

$$\Delta\tau_g = \frac{2\pi\alpha}{\omega}\left[\frac{\sin\pi\alpha}{\pi\alpha}\sin(\omega t - \Theta - \pi\alpha) - \sin(\omega t - \Theta - 2\pi\alpha)\right].$$

Finally, introducing expressions (4.77), (4.82) into integral (4.75) and retaining the first-order terms of smallness, we have

$$I_{\text{нав}}(t) = \frac{v_0}{g}\int_{\tau_g^0}^{t} I_\sim(\tau)\,d\tau + \frac{v_0}{g}\int_{\tau_g}^{\tau_g^0} I_\sim(\tau)\,d\tau + \varepsilon\frac{v_0}{g}\int_{\tau_g^0}^{t} I_\sim(\tau)\,u(t,\tau)\,d\tau. \tag{4.83}$$

First term does not depend on low parameter ε and gives zero approximation to the induced current

$$I^0(t) = \frac{v_0}{g}\int_{t-\frac{g}{v_0}}^{t} I_\sim(\tau)\,d\tau.$$

Second and third terms contain the members of first approximation. Zero approximation to a resonance harmonic of the induced current is equal

$$I^0(t) = \frac{2v_0 I_1}{\omega g}\sin\frac{\omega g}{2v_0}\cos\left(\omega t - \varphi_1 - \frac{\omega g}{2v_0}\right).$$

Value

$$\varphi_c = \varphi_1 + \omega \frac{g}{2v_0}$$

is the phase of the unloaded field in which the middle of cluster passes the center of the accelerating clearance. This phase differs from synchronous, since the latter is counted off from the maximum of the loaded field (Fig. 4.14):

$$\varphi_c = \varphi_s + \Theta.$$

Taking into account expression for I_1 (4.79) zero approximation is reduced to the form

$$I^0(t) = I_n \cos(\omega t - \varphi_c); \qquad (4.84)$$

$$I_n = 2I \frac{\sin|\varphi_s|}{\pi} \cdot \frac{\sin \pi\alpha}{\pi\alpha}. \qquad (4.85)$$

Let us determine the power, selected/taken by beam from the high-frequency field in the clearance. For this it suffices to use zero approximation to strength of induced current:

$$P_n = \frac{\omega}{2\pi} \int\limits_0^{2\pi/\omega} I^0(t) U_c(t)\, dt.$$

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After substituting into the integral of function (4.76), (4.84), we have

$$P_n = \frac{1}{2} I_n V \cos\varphi_s \qquad (4.86)$$

or, taking into account of expression (4.79), (4.85),

$$P_a = I_{cp} V T \cos\varphi_s \frac{\sin\varphi_s}{\varphi_s}. \qquad (4.87)$$

Parameter

$$T = \frac{\sin\pi\alpha}{\pi\alpha}$$

in equality (4.87) - the factor of transit time. If one considers that at the length of accelerator all particles, seized in acceleration mode, acquire on the average the same energy, as synchronous particle, then the power, selected/taken by beam from the high-frequency field, taking into account one clearance will comprise

$$P_n = I_{cp} V T \cos\varphi_s. \qquad (4.88)$$

Expression (4.87) differs from expression (4.88) in terms of factor $\frac{\sin\varphi_s}{\varphi_s}$, close to unity. This factor arose in connection with the fact that during the derivation of formula (4.87) is not considered mixing particles at the length of accelerator, connected with the longitudinal vibrations.

In order to obtain in sum (4.83) the resonance members of first approximation, it is necessary to utilize the second harmonic of beam current in expansion (4.80). The fundamental harmonic of beam current, as it is possible to be convinced, does not give the contribution to the resonance terms of first approximation. After the appropriate substitutions and integration from expression (4.83) it follows

$$I_{\text{нав}}(t)=I^{0}(t)-\frac{1}{2}\varepsilon I_{2}\left(1-\frac{\sin 2\pi\alpha}{2\pi\alpha}\right)\sin(\omega t-2\varphi_{c}+\Theta). \quad (4.89)$$

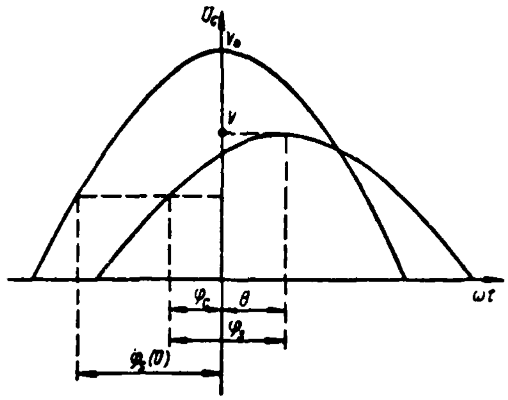

Fig. 4.14.

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After substituting into equality (4.89) for values ε and I_2 respectively of expression (4.78), (4.79) and after passing from the peak beam current to the average, we will obtain

$$I_{нав}(t)=I^0(t)+Y\left[V\sin 2\varphi_в\cos(\omega t-\Theta)-V\cos 2\varphi_в\sin(\omega t-\Theta)\right].$$

Value

$$Y=\frac{I_{cp}}{4\pi\alpha U_в}\left(1-\frac{\sin 2\pi\alpha}{2\pi\alpha}\right)\frac{\sin 2\varphi_в}{2\varphi_в} \qquad (4.90)$$

has a dimensionality of conductivity. According to expression (4.76),

$$V\cos(\omega t-\Theta)=U_c;$$
$$-V\sin(\omega t-\Theta)=\frac{1}{\omega}\cdot\frac{dU_c}{dt}.$$

Hence we have

$$I_{\text{нав}}(t) = I^0(t) - Y\left(\sin 2\varphi_s \cdot U_c + \frac{\cos 2\varphi_s}{\omega} \cdot \frac{dU_c}{dt}\right).$$

The parameters of beam $R_п$, $C_п$ are determined by expression (4.70). Thus,

$$\frac{1}{R_п} = Y \sin 2\varphi_s;$$
$$C_п = \frac{1}{\omega} Y \cos 2\varphi_s. \qquad (4.91)$$

Conductivity Y is proportional to the ratio of average/mean beam current to the energy of particles and, therefore, very low value. As shown below, usually $R_п \gg Z$. From equality (4.91) it is evident that value $\frac{1}{R_п}$ and $\omega C_п$ — one order. Therefore it is possible to disregard second term in the numerator of expression (4.72) and to obtain the following approximation formula for the relative frequency switch of the loaded duct/contour

$$\frac{\Delta\omega}{\omega} = -\frac{YZ}{2Q} \cos 2\varphi_s.$$

The beam, seized into acceleration mode, decreases the natural frequency of resonator. The equivalent energy factor of the loaded duct/contour is determined by formula (4.73), from which it follows

$$Q_{экв} = \frac{Q}{1 + YZ \sin 2\varphi_s}.$$

Let us note that quality $Q_{экв}$ does not determine total losses in the loaded duct/contour, but only that part of the losses which depends on voltage across capacitor and corresponds to linear processes in the duct/contour. Basic part of the losses, connected with the acceleration of beam, is determined by expression (4.86) and in the first approximation, does not depend on the amplitude of accelerating voltage, since under the assigned law of an increase in the lengths of drift tubes $V\cos\varphi_s = \text{const}$.

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Generally, the concept of the energy factor of oscillatory circuit regarding relates only to the linear processes. However, the losses of power for acceleration (4.86) lead to the fact that the loaded duct/contour proves to be nonlinear; so that this part of the losses cannot be, strictly speaking, expressed through the equivalent quality. In loaded mode/conditions $Q_{экв} > Q$, since $\varphi_s < 0$.

For the numerical estimation let us accept the following values of the parameters: λ=2m; $\cos\varphi_s = 0{,}8$; $\alpha = 0{,}25$; $U_s = 700$ keV; Q=80000; shunt-impedance for the unit of the length of resonator Z_1=20 MΩ/m. Then $Z=Z_1\beta\lambda=1.6\cdot10^{6}$ ohm. Let the average/mean beam current be equal to 100 mA. In this case of $Y=1.2\cdot10^{-6}\ \Omega^{-1}$; $\frac{\Delta\omega}{\omega} = -3{,}5\cdot10^{-6}$; $Q_{экв}/Q = 1{,}02$. Thus, the displacement of the natural frequency of resonator and change in its quality prove to be negligible. Frequency switch composes approximately $3\cdot10^{-3}$ widths of resonance band at the level 0.7. With an increase in the energy of the particles of the change in the natural frequency and quality decrease. Actually/really, since $U_c \doteqdot \beta^2$. that $Y=\beta^{-2}$. Shunt-impedance of section is proportional β. Consequently, $YZ=\beta^{-1}$.

From the given numerical estimations it is evident that in the ionic linear accelerators it is possible to disregard the members of first approximation, who are determining the dependence of the induced current on the stress/voltage in each accelerating clearance. Zero approximation to induced current (4.84), (4.85) sufficiently precise and can be used for the evaluations of effects, connected with the effect of beam on accelerating field.

Let us return to equation (4.71), after assuming $C_n = 0$; $\frac{1}{R_n} = 0$. Since the quality of the accelerating system usually is sufficiently high, let us disregard/neglect also the effect of the induced current, connected with the ohmic losses, assuming/setting

$$\frac{1}{Q} |I^0|_{\text{макс}} \ll \frac{1}{\omega} \left|\frac{dI^0}{dt}\right|_{\text{макс}}.$$

As a result we will obtain the following equation of the loaded duct/contour

$$\frac{d^2U_c}{dt^2} + \frac{\omega}{Q}\cdot\frac{dU_c}{dt} + \omega^2 U_c = \omega^2 \mathcal{E}(t) - \frac{\omega Z}{Q}\cdot\frac{dI^0}{dt}(t). \qquad (4.92)$$

In this case it is assumed that the natural frequency of the accelerating system coincides with the frequency of accelerating field.

If generator works in understressed mode/conditions [131], then equivalent emf, introduced into the duct/contour, in the first approximation, does not depend on the amplitude of voltage across capacitor. Regarding, steady voltage across capacitor of the unloaded duct/contour is equal

$$U_c^0(t) = V_0 \cos \omega t. \qquad (4.93)$$

Let us accept for simplicity, that the phase of current in the coupling loop does not depend on the parameters of field in the resonator.

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Equivalent emf, which corresponds to generator in the understressed mode/conditions, must take the form

$$\text{Э}(t) = -\frac{V_0}{Q} \sin \omega t. \qquad (4.94)$$

But if generator is found in the overstressed mode/conditions, then equality (4.94) remains valid only in the stationary unloaded state, and in the transient processes, by the specified load of duct/contour by beam, equivalent emf is changed depending on the current amplitude of voltage across capacitor. By analogy with equality (4.94) let us accept in the general case the following expression for equivalent emf, introduced into the duct/contour of the generator

$$\mathcal{E}(t)=-\frac{V_r}{Q}\sin\omega t. \qquad (4.95)$$

Then in understressed mode/conditions $V_r=V_0=\text{const}$, in the overstressed mode/conditions, according to expression (4.76), $V_r=V_r(V)$, moreover $V_r=V_0$ with $t\leq 0$. The moment/torque of time t=0 corresponds to the beginning of the injection of beam into the accelerator.

Let us substitute into the right side of equation (4.92) expressions (4.84), (4.95)

$$\frac{d^2U_c}{dt^2}+\frac{\omega}{Q}\cdot\frac{dU_c}{dt}+\omega^2U_c=\frac{\omega^2}{Q}\left[-V_r\sin\omega t+ZI_n\sin(\omega t-\varphi_c)\right]. \qquad (4.96)$$

Into equation (4.96) enters phase φ_c, with which the center of cluster is passed the middle of the accelerating clearance. With was examined the action of the induced current on the single duct/contour, this phase could be assigned arbitrarily. In the linear accelerator phase φ_c is not arbitrary. In order to cable phase φ_c to the stress/voltage, which acts in the clearance of this section, it is necessary to consider the character of the motion of clusters along the axis of entire accelerator. If beam does not undergo preliminary grouping, then clusters in the transient mode/conditions are formed/shaped around synchronous phase $\varphi_s(t)$. of that corresponding to this instantaneous value/significance of the amplitude of accelerating voltage V(t), which changes in the process of the establishment

$$V(t)\cos\varphi_s(t) = V_0\cos\varphi_s(0) = \text{const.} \qquad (4.97)$$

In this case coherent component of longitudinal vibrations is absent and the centers of clusters will pass the middles of all accelerating clearances to one and the same phase φ_c, of the equal to

$$\varphi_c(t) = \Theta(t) + \varphi_s(t). \qquad (4.98)$$

Phase φ_c in the process of establishment changes, in the first place, due to a change in the amplitude of accelerating voltage V(t), and, in the second place, due to the phase shift of this stress/voltage relative to stress/voltage in the unloaded resonator $\theta(t)$. But if beam undergoes preliminary grouping, then to determine phases $\varphi_c(t)$ for each clearance is complicated.

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Due to a change of the synchronous phase in the process of establishment appear coherent longitudinal vibrations. A change in the amplitude of the loaded field leads also to the frequency switch of longitudinal vibrations, and phase $\varphi_c(t)$ for each clearance proves to be the very complex function of longitudinal coordinate and time.

For simplification in the task we will consider that the beam does not pass buncher and clusters are formed/shaped in the linear accelerator. In this case let us assume that the particles, not seized into acceleration mode, do not interact with the field or they are lost sufficiently rapidly, so that in the first approximation, through each accelerating clearance passes only the bunched beam. Then for calculating the transient processes it is possible to use relationships/ratios (4.85), (4.98). The amplitude of induced current I_H (4.85) depends on synchronous phase and is function from the amplitude of accelerating voltage. Thus, parameters V_r, I_H, φ_c in the right side of equation (4.96) are connected with the amplitude of accelerating voltage with nonlinear dependences and they are also the slow functions of time.

The equation of establishment (4.96) can be simplified. We differentiate equality (4.76)

$$\frac{dU_c}{dt}=\frac{dV}{dt}\cos(\omega t-\Theta)+\left(\frac{d\Theta}{dt}-\omega\right)V\sin(\omega t-\Theta).$$

Since by equality (4.76) instead of one unknown function $U_c(t)$ are introduced two the unknowns V(t), Θ(t), on the variable/alternating V, Θ it is possible to apply one arbitrary condition which let us select as follows:

$$\frac{dV}{dt}\cos(\omega t-\Theta)+V\frac{d\Theta}{dt}\sin(\omega t-\Theta)=0. \qquad (4.99)$$

Then it remains

$$\frac{dU_c}{dt}=-\omega V\sin(\omega t-\Theta).$$

Let us substitute function U_c (4.76) in equation (4.96). Taking into account condition (4.99) we will obtain

$$-\frac{dV}{dt}\sin(\omega t-\Theta)+V\frac{d\Theta}{dt}\cos(\omega t-\Theta)=$$
$$=-\frac{\omega V_r}{Q}\sin\omega t+\frac{\omega}{Q}ZI_H\sin(\omega t-\varphi_c)+\frac{\omega}{Q}V\sin(\omega t-\Theta). \qquad (4.100)$$

Further, let us solve equations (4.99), (4.100) algebraic overally unknowns $\frac{dV}{dt}$, $V\frac{d\Theta}{dt}$. Let us arrive at the system of two first-order equations. This system, generally speaking, is not simpler than initial differential equation (4.96). However, now it is possible to use the "slowness" of functions V, Θ.

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Namely, let us integrate piecemeal system of equations of first-order, solved relative to derivatives, for the period of high frequency. All slow functions we will consider constants in the period $2\pi/\omega$ and let us remove them from under the integral signs. As a result we will obtain the following shortened equations of establishment [132, 23]:

$$\frac{dV}{dt}=\frac{\omega}{2Q}(V_{r}\cos\Theta-ZI_{H}\cos\varphi_{s}-V),$$
$$\frac{d\Theta}{dt}=-\frac{\omega}{2QV}(V_{r}\sin\Theta+ZI_{H}\sin\varphi_{s}). \quad (4.101)$$

When generator is disconnected from the duct/contour and the latter is not loaded, then

$$\frac{dV}{dt}=-\frac{\omega}{2Q}V.$$

Hence it is apparent that value

$$\tau_0=\frac{2Q}{\omega} \quad (4.102)$$

is time constant of a field slope in the unloaded accelerating system after the cutoff/disconnection of high-frequency oscillator.

Are of interest two limiting cases. If the duration of beam substantially exceeds the time of the establishment of field, then the acceleration of basic part of the beam occurs at the steady-state values of amplitude and phase of field, that correspond to the loaded mode/conditions. We deal, actually, concerning acceleration mode of steady beam. In this mode/conditions computed value of synchronous phase must be established/installed with the stationary amplitude of field in the resonator, loaded with beam. In the second case the duration of beam is considerably less than the time of the establishment of field. Acceleration occurs due to the energy, preliminarily accumulated in the resonator unless are taken special measures for power boosting of oscillators of the period of acceleration. The second case corresponds to the pulsed operation of accelerator and is utilized in essence in the linear accelerator-injectors. In the pulsed accelerators computed value of synchronous phase is established/installed with the amplitude of field, which corresponds to the unloaded resonator. Amplitude and phase of field for the time of the passage of pulse beam must remain within the limits of the assigned close tolerances.

Let us examine, first of all, steady loaded state. In steady state

$$\frac{dV}{dt}=0,\quad \frac{d\Theta}{dt}=0$$

and conservative values of amplitude and phase of accelerating voltage V, Θ satisfy the equations

$$\begin{aligned} V_r \sin\Theta &= -ZI_n \sin\varphi_s; \\ V_r \cos\Theta &= V + ZI_n \cos\varphi_s. \end{aligned} \qquad (4.103)$$

In these equations I_n, φ_s — conservative values of the amplitude of the induced current and synchronous phase.

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After dividing equations (4.103) on V and after considering relationships/ratios (4.66), (4.86), we will obtain

$$\frac{V_r}{V}\sin\Theta = -\frac{P_\text{п}}{P_\text{м}}\operatorname{tg}\varphi_s; \quad \frac{V_r}{V}\cos\Theta = 1 + \frac{P_\text{п}}{P_\text{м}}.$$

where $P_\text{п}$ — power, spent on the acceleration of the beam: $P_\text{м}$ — the power of high-frequency losses in copper in steady loaded state. Hence

$$\left(\frac{V_r}{V}\right)^2 = \left(\frac{P_\text{п}}{P_\text{м}}\right)^2 \operatorname{tg}^2\varphi_s + \left(1 + \frac{P_\text{п}}{P_\text{м}}\right)^2; \tag{4.104}$$

$$\operatorname{tg}\Theta = -\frac{P_\text{п}}{P_\text{п}+P_\text{м}}\operatorname{tg}\varphi_s. \tag{4.105}$$

Since $\varphi_s < 0$, that $\Theta > 0$; high-frequency field in the loaded resonator lags on the phase relative to field in the unloaded resonator. From equation (4.104) it follows that the stationary amplitude of stress/voltage in the loaded clearance V is unambiguously connected with relation $P_\text{п}/P_\text{м}$, since, according to relationships/ratios (4.97), $\operatorname{tg}\varphi_s$ — single-valued function V. Equalities (4.66), (4.86) give

$$\frac{P_\text{п}}{P_\text{м}} = \frac{2ZP_\text{п}}{V^2}, \tag{4.106}$$

or

$$\frac{P_\text{п}}{P_\text{м}} = \frac{Z}{V} I_\text{н}\cos\varphi_s.$$

Relation $P_\text{п}/P_\text{м}$ in each section of resonator in the first approximation, does not depend on energy of particles, in view of the fact that $Z \doteq \beta$; $V \doteq \beta$; $I_\text{н}$, as follows from formula (4.85), with an accuracy to the constancy of the factor of transit time it remains identical in all clearances. Thus, amplitude and phase shift of the loaded field is approximately identical in all sections of resonator. The load of high-frequency field with beam current does not lead to the "inclination/slope" of field, if only are retained the average/mean current of the accelerated beam and the linear value of shunt-impedance along the axis of resonator.

The obtained above stationary relationships/ratios remain valid and in the case of preliminary beam bunching. If beam passes through the buncher, then it is necessary that the clusters at the entrance of linear accelerator would be seized into acceleration mode in amplitude and phase of field, that correspond to the unloaded mode/conditions. From relationship/ratio (4.105) it follows that this condition is satisfied always. Actually/really, according to expression (4.105), with any value of load $\theta < \varphi_s$. Bunching parameters are such, that the centers of clusters pass the middle of the first clearance to phase φ_s relative to the loaded field; relative to the unloaded field (in the constant phase of the oscillations of stress/voltage in the buncher) the centers of clusters are passed to the phase

$$\varphi_c = \theta + \varphi_s < 0.$$

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This inequality provides the capture of clusters into acceleration

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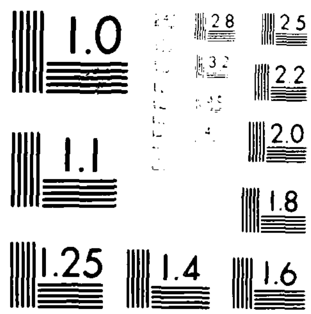

MICROCOPY RESOLUTION TEST CHART

mode in the unloaded field, since the right boundary of the region of capture exists $\varphi_{k} = \varphi_{s}$. If buncher is supplied by the high-frequency energy, directly abstracted/removed from the first resonator, then latter/last inequality is improved, since disappears parasitic phase displacement θ between the fields in buncher and resonator of accelerator.

Let us examine in more detail case $V_r = V_0 = \text{const}$. Let us substitute relation (4.106) in equation (4.104). Let us arrive at the biquadratic equation

$$V^4 - (V_0^2 - 4ZP_n)V^2 - \frac{4Z^2P_n^2}{\cos^2\varphi_s} = 0.$$

Hence

$$\left(\frac{V}{V_0}\right)^2 = \frac{1}{2} - \frac{2ZP_n}{V_0^2} \pm \sqrt{\left(\frac{1}{2} - \frac{2ZP_n}{V_0^2}\right)^2 - \left(\frac{2ZP_n}{V_0^2\cos\varphi_s}\right)^2}. \quad (4.107)$$

Thus, if carried out is the condition

$$\frac{V_0^2}{2} - 2ZP_n > \frac{2ZP_n}{\cos\varphi_s}, \quad (4.108)$$

then there are two values for the stationary amplitude. The presence of two values for the stationary amplitude in the loaded mode/conditions is explained by the fact that the power, taken away in field by the beam, seized into the acceleration, does not depend on the amplitude of field; therefore the loaded system proves to be nonlinear. As shown subsequently, the upper value/significance of amplitude is stable, and lower is unstable. In proportion to an increase in value $\frac{2ZP_n}{V_0^2}$ both roots (4.107) converge, moreover stable amplitude decreases, and unstable increases. With

$$\frac{V_0^2}{2} - 2ZP_n = \frac{2ZP_n}{\cos\varphi_s} \qquad (4.109)$$

both amplitudes they pour at metastable point, and biquadratic equation has the multiple root, which corresponds

$$\left(\frac{V}{V_0}\right)^2 = \frac{1}{2(1+\cos\varphi_s)}.$$

Thus, any stable stationary amplitudes satisfy the condition

$$\left(\frac{V_0}{V}\right)^2 < 2(1+\cos\varphi_s). \qquad (4.110)$$

But, according to expression (4.106),

$$\frac{P_n}{P_M} = \frac{2ZP_n}{V_0^2}\left(\frac{V_0}{V}\right)^2.$$

Taking into account expressions (4.109), (4.110) we have

$$\frac{P_n}{P_M} < \cos\varphi_s. \qquad (4.111)$$

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This relationship/ratio between the power, spent on the acceleration, and the power of losses in copper determines the condition for existence of stable stationary amplitude in the resonator, loaded with the steady beam (during the understressed mode/conditions oscillator). From formula (4.105) and condition (4.111) it follows

$$\operatorname{tg}\Theta < \operatorname{tg}\frac{\varphi_s}{2},$$

so that phase displacement of the loaded field of that of relatively unloaded does not exceed $\frac{1}{2}\varphi_s$. Latter/last inequality substantially more precisely formulates formula (4.105), but it is correct only when V_r, I_n const.

Upon the acceleration of the beam of high intensity condition (4.111) can prove to be disturbed. In that case stable stationary field will exist only during the overstressed mode/conditions of oscillator.

The obtained results, which relate to the stationary loaded field, have simple geometric interpretation. Let us multiply the first equation of establishment (4.101) by the instantaneous value of the established/installed amplitude V

$$\frac{1}{2}\frac{d}{dt}V^2 = \frac{\omega}{2Q}\{V_r V\cos\Theta - ZI_n V\cos\varphi_s - V^2\}. \qquad (4.112)$$

But

$$W_э = \frac{CV^2}{2}$$

is average in the period the value/significance of the energy, accumulated in the electric field, and

$$P_n = \frac{1}{2}I_n V\cos\varphi_s$$

there is the power, spent on the acceleration of beam. Let us replace the instantaneous value of the amplitude of stress/voltage in equation (4.112) with the medium energy, inclined in accelerating field, and will place cos $\theta \simeq 1$. After considering relationships/ratios (4.67) and after designating

$$A_r = V_r \sqrt{\frac{\omega}{8QZ}},$$

we will obtain

$$\frac{dW_э}{dt} = A_r \sqrt{W_э} - \frac{\omega}{Q} W_э - P_п. \qquad (4.113)$$

Formula (4.113) is the equation of the energy balance in the resonator. On the left side stands rate of change in the energy of accelerating field, while in the right - difference between the power, given up by the oscillator

$$P_r = A_r \sqrt{W_э},$$

and the sum of the losses to the acceleration and of losses in copper. The process of establishment is conveniently examined on plane $W_э$, $\frac{dW_э}{dt}$.

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Fig. 4.15 gives the diagrams of the establishment of field. Straight lines correspond to the law of an increase in the high-frequency losses, parabola - to law of an increase in the power, given up by oscillator in the case $A_r = \text{const}$ (understressed mode/conditions). The abscissa of the point of intersection of parabola with straight line 1 is equal to the steady-state value of the medium energy of unloaded field $W_э^0$. The load, introduced by beam, leads to the parallel displacement by the straight line of high-frequency losses upward. To case (4.108) correspond straight/direct 2; it is easy to see that the upper state of equilibrium is stable, and lower is unstable. If oscillator does not cover/coat total high-frequency losses, then condition (4.108) is not satisfied and the steady states of field are absent (straight line 3). For the compensation for total power losses of oscillator must be raised; parameter A_r in this case grows/rises. However, if losses to the acceleration are too great, so that condition (4.111) is not satisfied, then the assigned stationary field will prove to be unstable (parabola, carried out by dotted line). The stability of the assigned stationary field can be then it is provided only in the overstressed mode/conditions (dot-dash curve).

Let us examine the pulsed mode of accelerator. For the analysis of pulsed operation more conveniently to switch over to the equation of the establishment of field. The medium energy, accumulated in accelerating field, is connected with the amplitude of field on the axis of accelerator with the general/common/total relationship/ratio

$$W_э = a_э E^2. \qquad (4.114)$$

Coefficient $a_э$ is determined by the frequency of accelerating field, by transmission mode and by geometry of the accelerating devices/equipment, and also by parameters of dielectric, if the latter is in the resonance volume of the accelerating elements/cells.

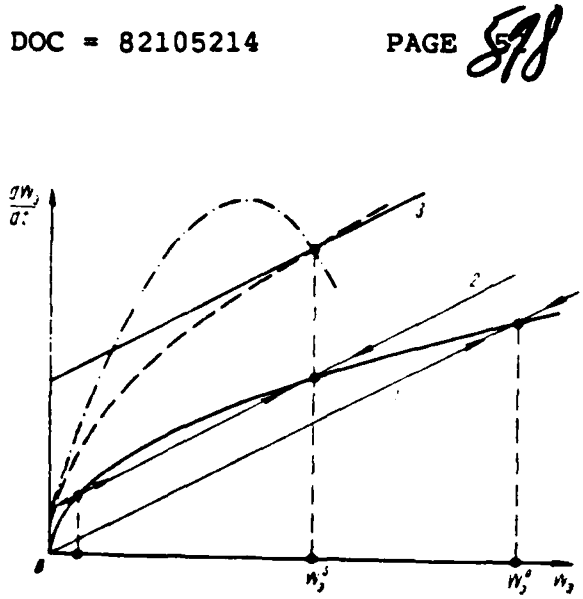

Fig. 4.15.

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In view $P_n = \text{const}$ the equation of the establishment of field is nonlinear:

$$\frac{dE}{dt} = \frac{A_r(E)}{2a_э} - \frac{\omega}{2Q} E - \frac{P_n}{2a_э^2} \cdot \frac{1}{E} \cdot \qquad (4.115)$$

Let prior to the beginning of injection in the axis of resonator be established stationary unloaded field $E = E_0$. This value/significance corresponds to the operational conditions of acceleration. Upon the injection of beam the equilibrium is disrupted; however, for the transit time of the beam with a duration of τ_n a relative field slope

$$x(t) = \frac{E(t) - E_0}{E_0}$$

must remain within the limits allowances for the nominal amplitude.

Since $|x|<<1$, we can be restricted to the linear section of the full-load saturation curve of oscillator near the operating point

$$A_r(E)=A_r(E_s)-\frac{dA_r}{dE}(E_s)E_s x.$$

After assuming $|x|<<1$, we will obtain the following equation for a relative field slope

$$\frac{dx}{dt}=-\frac{1}{\tau_0}\left(1-\nu_r-\frac{P_n}{P_M}\right)x-\frac{P_n}{\tau_0 P_M}. \qquad (4.116)$$

where τ_0 - time constant of a field slope (4.102), and dimensionless parameter ν_r is equal to

$$\nu_r=-\frac{Q}{\omega u_3}\cdot\frac{dA_r}{dE}(E_s)$$

and it is determined by the inclination/slope of the full-load saturation curve of oscillator. In the understressed mode/conditions of oscillator $\nu_r=0$; in overstressed mode/conditions $\nu_r>0$. moreover, as a rule $\nu_r<1$. Since x(0)=0, the equation (4.116) has the following solution

$$x(t)=-\frac{P_n P_M}{1-\nu_r-P_n/P_M}\left[1-e^{-\frac{t}{\tau_0}\left(1-\nu_r-\frac{P_n}{P_M}\right)}\right]. \qquad (4.117)$$

A field slope near the working value depends substantially on the full-load saturation curve of oscillator.

Let $\nu_r=0$. Then when $\tau_n\ll\tau_0$ we have

$$x(t)\approx-\frac{P_n}{\tau_0 P_M}t.$$

But, according to expressions (4.65), (4.102)

$$P_{м}\tau_0 = 2W_э.$$

On the other hand,

$$P_{п}\tau_{п} = W_{п},$$

where $W_{п}$ – energy, selected/taken by beam from accelerating field.

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Therefore toward the end of the impulse/momentum/pulse of ion current a relative field slope comprises

$$\frac{\Delta E}{E_{a}} = \frac{W_{п}}{2W_{э}}. \qquad (4.118)$$

The ratio of the energy, selected/taken by beam, to the medium energy, accumulated in accelerating field, must not exceed double the allowance for the nominal value of accelerating field.

Let now $\nu_r > 0$. If ν_r is small and the condition

$$\tau_{п} \ll \frac{\tau_0}{1+\nu_r-\frac{P_{п}}{P_{м}}}$$

is satisfied, then remains valid relationship/ratio (4.118). But if $\nu_r \gg 1$, then a relative field slope remains small independent of the duration of beam and comprises

$$\frac{\Delta E}{E_{a}} \approx \frac{P_{п}}{\nu_r P_{м}}.$$

In this case it is assumed that power loss for the acceleration of the same order as high-frequency power losses in copper or less than the latter. The case of a deep cooling of the accelerating system requires separate examination.

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